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Poincaré group and relativistic wave equations in $2+1$ dimensions

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Abstract. Using the generalized regular representation, an explicit construction of the unitary irreducible representations of the $(2+1)$-Poincaré group is presented. A detailed description of the angular momentum and spin in $2+1$ dimensions is given. On this base the relativistic wave equations for all spins (including fractional) are constructed.

1. Introduction

At the present time a great deal of attention is being devoted to field theoretical models in $(2+1)$-dimensional spacetime [1]. There is a possibility that particles exist with fractional spin and exotic statistics in this space. These particles, which are called anyons, may have a relation to the physics of planar phenomena, for example, to the fractional quantum Hall effect [2].

The corresponding Poincaré group, which will be denoted as $M(2,1)$, was studied in [3] and from the the field theoretical point of view in [4]. The importance of the investigation of the $M(2,1)$ group is also stressed by the fact that, being a subgroup of the Poincare group in $3+1$ dimensions $M(3,1)$, it retains many of the properties of the latter. In this connection, some of the results, which can be derived for the $M(2,1)$ group, may also be valid for the $M(3,1)$ group. It should be remarked that in contrast to $M(1,1)$, discussed in detail in [5], $M(2,1)$ has a non-Abelian and non-compact subgroup of rotations, similar to $M(3,1)$, that leads to a non-trivial structure of the spinning space.

The aim of the present work is to construct a detailed theory of the $M(2,1)$ group representations in a form which may be convenient for physical applications. Namely, we try to emphasize the problem of the spin description and the construction of relativistic wave equations.

In the seminal paper [6] Wigner gave a classification of all unitary irreductible representations of the $3+1$ Poincaré group, together with a prescription for their explicit constraction. In original papers [7] using the Wigner prescription, the unitary irreductible representations of $M(3,1)$ were explicitly determined and a synthesis of covariant partical equations connected with this representation was carried out. This approach to the representation theory of $M(3,1)$ has been discussed in detail in numerous papers and books [8–12]. On the other hand, there is in fact only one work [3] where the representation theory of $M(2,1)$ has been studied directly. Thus, we hope that the present paper can add some important details to the latter theory.

When classifying the representations of semi-direct products, one usually uses the method of the little group [6]. That method was also applied to $M(2,1)$ in [3]. However, for our purposes of the detailed and explicit construction of representations it is more convenient to use both the little group method and the method of harmonic analysis and, in particular, the generalized regular representation (GRR). It is known that any irreducible representation (IR) of a Lie group is equivalent to a sub-representation of the left (right) GRR [13–15]. The harmonic analysis allows the most complete description to be given of representations of a group Lie, using explicit realizations in spaces of functions on the group. The ideas of the method are presented for example in [11], where one can also find its application to the motion group of the plane $M(2)$. The harmonic analysis for the $M(3,1)$ group can be found in [18, 19]. The harmonic analysis is also very useful in the study of special functions properties (see [15, 16] and the Wigner lectures, produced by Talman [17]).

In the present work we use the quasi-regular and generalized regular representations to explicitly construct all unitary IRs of $M(2,1)$ and to analyse on this basis the relativistic wave equations for higher spins (including fractional) and the corresponding coherent states. Studying the quasi-regular representation of $M(2,1)$, we introduce the scalar fields and construct the relativistic theory of $2+1$ angular orbital momentum. Presenting $(2+1)$-dimensional vectors by means of 2×2 matrices, we introduce a parametrization of the $M(2,1)$ group, where the rotations are given by two complex numbers z_1 and z_2, $|z_1|^2 - |z_2|^2 = 1$, which are analogues of Cayley–Klein parameters of the compact case. The representation space of the left GRR consists of scalar functions $f(x,z)$, whereas the spinning operators can be presented as first-order differential operators in the variables z. It is convenient to classify representations not only with respect to the Casimir operators $\hat{\boldsymbol{p}}^2 = p_\mu p^\mu$ and $\hat{W} = \hat{p}_\mu \hat{J}^\mu$, but also with respect to the operator of the square of the spin, which commutes with all generators of the left GRR. The latter operator marks representations of the $2+1$ Lorentz group.

In the framework of such an approach one can naturally construct relativistic wave equations for particles with arbitrary spin. The fixation of the value of the square of the spin $S(S+1)$ defines the structure of the z-dependence of the functions $f(x,z)$, namely, they appear to be (quasi-)polynomials of the power $2S$ on z. The coefficients of these polynomials are interpreted as components of finite(infinite)-dimensional wavefunctions of relativistic particles with higher spins. The fixation of the values of the Casimir operators provides equations for these components.

In such a way, for example, both $2+1$ Dirac equation (equation for spin 1) and equations for particles with fractional spins, which are related to the discrete series of the Lorentz group (see [4, 20, 21]) appear. Thus, using GRR one achieves a unique approach with which to describe particles with different spins and also provides a possibility to establish a relation between different descriptions of these spins, for example, in terms of scalar functions $f(x,z)$ or in terms of multicomponent columns $\psi(x)$.

A detailed description of angular momentum and spin in $2+1$ dimensions is given on the base of the representation theory of $SU(1,1)$, which is summarized in the appendix. In particular, multivalued unitary IRs of $SO(2,1) \sim SU(1,1)$ and corresponding coherent states (CS) are considered. It is interesting to discover that the $2+1$ Dirac equation also appears in the latter case as an equation for CS evolution.

The $SO(2,1)$ group appears not only in particle physics but has many other physical applications. For example in the classical theory of light propagation [22] and especially in quantum optics where this group is useful for the description of the coherent and squeezed states of light [23]. The coherent and squeezed states are canonically transformed states of the groundstate harmonic oscillator. A subset of these transformations form a group Sp(2) which is locally isomorphic to the $(2+1)$-dimension Lorentz group.

2. Parametrization

$M(2,1)$ is a six-parametric group of motions of (2+1)-dimensional pseudo-Euclidean space, it preserves the interval $\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$, where $x = (x^\mu)$, $\mu = 0, 1, 2$, are coordinates, and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$ is the Minkowski tensor. The transformation of the vector x under the action of the group (vector representation) is given by the formula

$$x' = gx \qquad g \in M(2,1) \qquad x'^\nu = \Lambda^\nu_\mu x^\mu + a^\nu \tag{2.1}$$

where Λ is a 3×3 rotation matrix of the $2+1$ Lorentz group $O(2,1)$. The transformations can also be presented in the four-dimensional form,

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ 1 \end{pmatrix} = \begin{pmatrix} & & & a^0 \\ & \Lambda(\alpha) & & a^1 \\ & & & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ 1 \end{pmatrix} \tag{2.2}$$

with the composition low $(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2\Lambda_1)$. The latter means that $M(2,1)$ is the semi-direct product of the $2+1$ translation group $T(3)$ and the Lorentz group $O(2,1)$,

$$M(2,1) = T(3) \times) O(2,1).$$

As is known the group $O(2,1)$ contains four disjoint sets $O^\uparrow_+$ ($\det\Lambda = +1$, $\Lambda^0_0 > 0$), $O^\downarrow_+$ ($\det\Lambda = +1$, $\Lambda^0_0 < 0$), $O^\uparrow_-$ ($\det\Lambda = -1$, $\Lambda^0_0 > 0$), $O^\downarrow_-$ ($\det\Lambda = -1$, $\Lambda^0_0 < 0$), where only $O^\uparrow_+ = SO_0(2,1)$ is connected to the identity continuously. The two sets $O^{\uparrow\downarrow}_+$ are equivalent to the group $SO(2,1)$. The corresponding continuously connected part of $M(2,1)$ is $T(3)\times)SO_0(2,1)$.

Consider first the group $SO_0(2,1)$. One-parametrical subgroups of $SO_0(2,1)$, which correspond to the rotations around axes x^0, x^1, x^2, are given by the matrices

$$\Lambda_{x^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_0 & -\sin\alpha_0 \\ 0 & \sin\alpha_0 & \cos\alpha_0 \end{pmatrix} \qquad \Lambda_{x^1} = \begin{pmatrix} \cosh\alpha_1 & 0 & \sinh\alpha_1 \\ 0 & 1 & 0 \\ \sinh\alpha_1 & 0 & \cosh\alpha_1 \end{pmatrix}$$
$$\Lambda_{x^2} = \begin{pmatrix} \cosh\alpha_2 & -\sinh\alpha_2 & 0 \\ -\sinh\alpha_2 & \cosh\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

The general transformation can be written in the form $\Lambda_{x^\mu} = \exp(-\mathrm{i}\alpha_\mu J^\mu)$, where the generators $J^\mu = \mathrm{i}(\mathrm{d}/\mathrm{d}\alpha_\mu)(\Lambda_{x^\mu})|_{\alpha=0}$ are

$$J^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \qquad J^1 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

They obey the commutation relations

$$[J^\mu, J^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta} J_\eta$$

where $\epsilon^{\mu\nu\eta}$ is the totally antisymmetric Levi-Civita symbol, $\epsilon^{012} = 1$.

It is also possible to write the finite transformations by means of $SL(2,R)$ matrices [3] or $SU(1,1)$ matrices. We will consider the latter possibility in detail, taking into account that $SO_0(2,1)$ is equivalent to $SU(1,1)/Z_2$, $Z_2 = \{I, -I\}$, where Z_2 is a multiplicative group consisting of two elements, I is the unit matrix. Thus, we are going to study the group $\tilde{M}(2,1) = T(3)\times)SU(1,1)$. The classification and construction of representations of $\tilde{M}(2,1)$ allow representations of the group $M(2,1)$ to be described.

There is a one-to-one correspondence between the $2+1$ Lorentz vectors x^μ and 2×2 matrices X. Let σ^0 be the unit 2×2 matrix and σ^1, σ^2 the two first Pauli matrices. Then

$$X = x^\mu\sigma^\mu = \begin{pmatrix} x^0 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 \end{pmatrix} \qquad \det X = X^2 = x_\mu x^\mu \qquad x^\mu = \tfrac{1}{2}\mathrm{Tr}(X\sigma^\mu). \tag{2.5}$$

The transformation (2.1) can be written in terms of the matrix representation in the form

$$X' = UXU^\dagger + A \tag{2.6}$$

where the matrices X', X, A correspond to the vectors x'^μ, x^μ, a^μ, and the $SU(1,1)$ matrix U,

$$\begin{aligned} &U = \begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} \bar{u}_1 & u_2 \\ \bar{u}_2 & u_1 \end{pmatrix} \\ &|u_1|^2 - |u_2|^2 = 1 \qquad u_1 = \cosh(\theta/2)\,\mathrm{e}^{\mathrm{i}(-\phi-\omega)/2} \qquad u_2 = -\sinh(\theta/2)\,\mathrm{e}^{\mathrm{i}(-\phi+\omega)/2} \\ &0 \leqslant \theta < \infty \qquad -2\pi \leqslant \phi < 2\pi \qquad 0 \leqslant \omega < 2\pi \end{aligned} \tag{2.7}$$

provides the Lorentz rotations. Its relation with the matrix Λ from $SO_0(2,1)$ is given by the formula

$$\Lambda = \begin{pmatrix} u_1\bar{u}_1 + u_2\bar{u}_2 & 2\,\mathrm{Re}(u_1\bar{u}_2) & 2\,\mathrm{Im}(u_1\bar{u}_2) \\ 2\,\mathrm{Re}(u_1u_2) & \mathrm{Re}(u_1^2 + u_2^2) & \mathrm{Im}(u_1^2 - u_2^2) \\ -2\,\mathrm{Im}(u_1u_2) & -\mathrm{Im}(u_1^2 + u_2^2) & \mathrm{Re}(u_1^2 - u_2^2) \end{pmatrix}.$$

One can remark that U and $-U$ correspond to one and the same Λ, so that to parametrize the rotations it is enough to use $\phi \in [0, 2\pi]$.

In the representation (2.6) u_1 and u_2 are analogues of Cayley–Klein parameters, and ϕ, θ, ω are those of the Euler angles, $U = U(\phi,\theta,\omega)$. It is possible to see that the matrices $U(\phi,0,0)$ and $U(0,0,\omega)$ correspond to the rotations around the axis x^0, $U(0,\theta,0)$ correspond to the rotations around the axis x^2 and $U(\phi,\theta,\omega) = U(\phi,0,0)U(0,\theta,0)U(0,0,\omega)$, i.e. the general transformation can be presented as the ω-rotation around the axis x^0, then the θ-rotation around the axis x^2, and again the ϕ-rotation around the axis x^0.

The following sets of the parameters (ϕ,θ,ω): $(\alpha_0,0,0)$, $(-\pi/2,\alpha_2,\pi/2)$, $(0,\alpha_1,0)$, correspond to the one parametrical subgroups $\Lambda_{x^0}(\alpha_0)$, $\Lambda_{x^1}(\alpha_1)$, $\Lambda_{x^2}(\alpha_2)$ respectively. The matrix Λ in the Euler angles parametrization can be presented as $\Lambda(\phi,\theta,\omega) = \Lambda_{x^0}(\phi)\Lambda_{x^2}(\theta)\Lambda_{x^0}(\omega)$.

We are also going to use the latter parametrization of elements g of $\tilde{M}(2,1)$ by means of matrices A and $SU(1,1)$ matrices U, $g = (A, U)$. In this representation the composition low and inverse elements have the form

$$\begin{aligned} &g = (A, U) = (A_2, U_2)(A_1, U_1) = (U_2A_1U_2^\dagger + A_2, U_2U_1) \\ &g^{-1} = (-U^{-1}A(U^{-1})^\dagger, U^{-1}). \end{aligned} \tag{2.8}$$

3. Quasi-regular representation and theory of orbital momentum

3.1. Quasi-regular representation and scalar field

Let us consider a quasi-regular representation $T(g)$, which is acts on the coset space $M(2,1)/O(2,1) = \tilde{M}(2,1)/SU(1,1)$, i.e. in the space of functions $f(x)$,

$$f'(x) = T(g)f(x) = f(g^{-1}x). \tag{3.1}$$

The representation (3.1) corresponds to a scalar field transformation low, $f'(gx) = f'(x') = f(x)$. The explicit form of $g^{-1}x$ is given by the formulae

$$(g^{-1}x)^\nu = (\Lambda^{-1})^\nu_\mu (x^\mu - a^\mu) \qquad g^{-1}x = U^{-1}(X - A)(U^{-1})^\dagger \tag{3.2}$$

in the parametrizations (2.1) and (2.6), respectively. The Lie algebra of $M(2,1)$ contains six generators $\hat{p}_\mu$ and $\hat{L}^\mu$, which correspond to the parameters a^μ and $-\alpha_\mu$. They have a form

$$\hat{p}_\mu = \mathrm{i}\partial/\partial x^\mu \qquad \hat{L}^\eta = \epsilon^{\eta\mu\nu}\hat{x}_\mu\hat{p}_\nu = \mathrm{i}\epsilon^{\eta\mu\nu}x_\mu\partial/\partial x^\nu \tag{3.3}$$

in the representation in question, and obey the commutation relations

$$[\hat{p}_\mu, \hat{p}_\nu] = 0 \qquad [\hat{p}^\mu, \hat{L}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{p}_\eta \qquad [\hat{L}^\mu, \hat{L}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{L}_\eta. \tag{3.4}$$

Finite transformations in the parametrizations (2.4) and (2.7) can be written as

$$T(g)f(x) = \mathrm{e}^{-\mathrm{i}\phi\hat{L}^0}\,\mathrm{e}^{-\mathrm{i}\theta\hat{L}^2}\,\mathrm{e}^{-\mathrm{i}\omega\hat{L}^0}\,\mathrm{e}^{\mathrm{i}a\hat{p}}f(x). \tag{3.5}$$

The eigenvalue m^2 of the Casimir operator† $\hat{\boldsymbol{p}}^2$ can, in particular, characterize the IR, $\hat{\boldsymbol{p}}^2 f_m(x) = m^2 f_m(x)$. For unitary representations, where the generators $\hat{p}_\mu$ and $\hat{L}^\mu$ are Hermitian, m^2 is real. It follows from the commutation relations (3.4) that $\hat{p}\hat{L}$ is also a Casimir operator, which is, however, zero in the representation under consideration.

To find all IRs, which are contained in the representation (3.1), we consider the space of functions which are dependent on momenta, doing the Fourier transformation,

$$\varphi(p) = (2\pi)^{-3/2}\int f(x)\,\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}x. \tag{3.6}$$

In this space the expressions for the generators have the form

$$\hat{p}_\mu = p_\mu \qquad \hat{L}^\eta = \epsilon^{\eta\mu\nu}\hat{x}_\mu p_\nu = \mathrm{i}\epsilon^{\eta\mu\nu}p_\mu\partial/\partial p^\nu. \tag{3.7}$$

The form of $\hat{L}^\mu$ in the space of functions $\varphi(p)$ coincides with that in the space of functions $f(x)$ if one replaces $p^\mu \to x^\mu$, and, therefore, the rotations result in: $\varphi(p) \to \varphi(p')$, where $p'_\mu = (\Lambda^{-1})^\nu_\mu p_\nu$. In the parametrization (2.6),

$$P' = U^{-1}P(U^{-1})^\dagger \qquad P = p^0 I + p^1\sigma^2 + p^2\sigma^2. \tag{3.8}$$

Translations affect only the phase of the functions, so we get an analogue of equation (3.1),

$$T(g)\varphi(p) = \mathrm{e}^{\mathrm{i}ap'}\varphi(p'). \tag{3.9}$$

IRs are related to orbits in the space of functions $\varphi(p)$ and are marked by the values $p^2 = (p')^2 = m^2$. We denote by $T_m(g)$ representations with a given m. We will consider three possible cases.

(1) $m \neq 0$ and is real. In this case the representations $T_m(g)$ act in the space of functions on a two-sheeted hyperboloid,

$$p_0 = \pm m\cosh\theta \qquad p_1 = \mp m\sinh\theta\cos\phi \qquad p_2 = \mp m\sinh\theta\sin\phi. \tag{3.10}$$

† Here and in what follows $\hat{\boldsymbol{p}}^2 = \hat{p}_\mu\hat{p}^\mu$ and so on.

At $m > 0$ it is decomposed in two IRs, one $T_m^+(g)$, which corresponds to particles (upper sheet, $p_0 > 0$), and another one $T_m^-(g)$, which corresponds to antiparticles (lower sheet, $p_0 < 0$). One can consider only IR with $m > 0$ because of $T_m^+(g)$ and $T_{-m}^-(g)$ are equivalent. The scalar product at a fixed m is given by the equation

$$\langle f_1|f_2\rangle = \int_0^{2\pi} \mathrm{d}\phi \int_0^{+\infty} \overline{\varphi_1(\theta,\phi)}\varphi_2(\theta,\phi)\sinh\theta\,\mathrm{d}\theta \tag{3.11}$$

and the generators L^μ have the form

$$\hat{L}^0 = -\mathrm{i}\partial_\phi \qquad \hat{L}^1 = -\mathrm{i}(\coth\theta\cos\phi\partial_\phi + \sin\phi\partial_\theta)$$
$$\hat{L}^2 = \mathrm{i}(-\coth\theta\sin\phi\partial_\phi + \cos\phi\partial_\theta). \tag{3.12}$$

(2) $m = 0$. In this case the representations $T_m(g)$ act in the space of functions on the cone,

$$p_0 = p \qquad p_1 = -p\cos\phi \qquad p_2 = -p\sin\phi. \tag{3.13}$$

The representation $T_0(g)$ is split into three IRs: one-dimensional $T_0^0(g)$, which corresponds to the invariant $p = 0$ (vertex of the cone), and $T_0^+(g)$ and $T_0^-(g)$, which act on the upper and lower sheets of the cone. The scalar product is given by the formula

$$\langle f_1|f_2\rangle = \int_0^{2\pi} \mathrm{d}\phi \int_0^{+\infty} \overline{\varphi_1(p,\phi)}\varphi_2(p,\phi)\,\mathrm{d}p \tag{3.14}$$

and the generators L^μ have the form

$$\hat{L}^0 = -\mathrm{i}\partial_\phi \qquad \hat{L}^1 = \mathrm{i}(\cos\phi\partial_\phi + p\sin\phi\partial_p) \qquad \hat{L}^2 = \mathrm{i}(-\sin\phi\partial_\phi + p\cos\phi\partial_p). \tag{3.15}$$

(3) m is imaginary, which corresponds to tachyons. The representations $T_m(g)$ act in the space of functions on a one-sheeted hyperboloid,

$$p_0 = \mathrm{i}m\sinh\theta \qquad p_1 = -\mathrm{i}m\cosh\theta\cos\phi \qquad p_2 = -\mathrm{i}m\cosh\theta\sin\phi. \tag{3.16}$$

The scalar product is given by the formula

$$\langle f_1|f_2\rangle = \int_0^{2\pi} \mathrm{d}\phi \int_0^{+\infty} \overline{\varphi_1(\theta,\phi)}\varphi_2(\theta,\phi)\cosh\theta\,\mathrm{d}\theta \tag{3.17}$$

and the generators L^μ have the form

$$\hat{L}^0 = -\mathrm{i}\partial_\phi \qquad \hat{L}^1 = -\mathrm{i}(\tanh\theta\cos\phi\partial_\phi + \sin\phi\partial_\theta)$$
$$\hat{L}^2 = \mathrm{i}(-\tanh\theta\sin\phi\partial_\phi + \cos\phi\partial_\theta). \tag{3.18}$$

3.2. Angular momentum

We have considered three types of scalar representations of $\tilde{M}(2,1)$, which correspond to a real mass, zero mass and imaginary mass. In each case the functional representation spaces are different, these are functions on one- or two-sheeted hyperboloids and on the cone. Respectively, the expressions for the angular momentum operators $\hat{L}^\mu$ are different. Here we are going to analyse the eigenvalue problem for the square of this operator and its projection in all the cases, using the p-representation (3.6) and the consideration given in the appendix. In particular, we will use bases of unitary IRs $SO(2,1)$ to decompose functions on one- and two-sheeted hyperboloids and on the cone.

(1) $m \neq 0$ and is real. The operators $\hat{L}^{\mu}$ act in the space of functions on two-sheeted hyperboloids with the scalar product (3.11). The rising and lowering operators $\hat{L}_{\pm}$ and the operator of the square of the angular momentum $\hat{\boldsymbol{L}}^2$ have the form

$$\hat{L}_{\pm} = \mathrm{e}^{\pm \mathrm{i}l\phi}(\pm \mathrm{i}\coth\theta\,\partial/\partial\phi + \partial/\partial\theta) \qquad \hat{\boldsymbol{L}}^2 = \frac{\partial^2}{\partial\theta^2} + \coth\theta\frac{\partial}{\partial\theta} + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{3.19}$$

Let us suppose that a representation of the $SO(2,1)$ subgroup has the highest weight $f(\theta, j)\,\mathrm{e}^{\mathrm{i}j\phi}$. Then

$$\hat{L}_{+} f_j(\theta)\,\mathrm{e}^{\mathrm{i}j\phi} = \mathrm{e}^{\mathrm{i}(j+1)\phi}(-j\coth\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0 \tag{3.20}$$

and therefore, the highest weight has the form $(\sinh\theta)^j\,\mathrm{e}^{\mathrm{i}j\phi}$. It is easy to remark that at $j < -1/2$ (that would correspond to a discrete series) the norm of the state has a power divergence as a result of a singularity at $\theta = 0$, and at $j > -1/2$ the integrand of the norm grows exponentially with the growth of θ (the case of double-valued IRs with $j = -1/2$ is considered below). That means that single-valued unitary IRs with a highest (lowest) weight are absent in the decomposition of $T_m^{\pm}$.

In the general case the wavefunction (3.5) in the p-representation, which are eigenvectors of the operators $\hat{\boldsymbol{L}}^2$, $\hat{L}^0$,

$$\hat{\boldsymbol{L}}^2|jl\rangle = j(j+1)|jl\rangle \qquad \hat{L}^0|jl\rangle = l|jl\rangle \tag{3.21}$$

can be written in the form $NP_j^l(\cosh\theta)\,\mathrm{e}^{\mathrm{i}l\phi}$, where $P_j^l(\cosh\theta)$ is adjoint Legendre function and N does not depend on θ and ϕ. We are going to use the functions $\tilde{P}_j^l(\cosh\theta) = (\Gamma(j+1)/\Gamma(j+l+1))P_j^l(\cosh\theta))$. The representation is unitary and single-valued at $j = -1/2 + \mathrm{i}\lambda/2$ and integer l (see [15]). Thus, IRs $T_m^{\pm}$ of $\tilde{M}(2,1)$ are decomposed in the course of the reduction into the representations of the principal series,

$$|\lambda l\rangle = \tilde{P}_j^l(\cosh\theta)\,\mathrm{e}^{\mathrm{i}l\phi}/\sqrt{2\pi} \qquad j = -1/2 + \mathrm{i}\lambda/2 \tag{3.22}$$

$$\langle\lambda l \mid \lambda' l'\rangle = (1/2\pi^2)\lambda\tanh(\pi\lambda/2)\delta(\lambda-\lambda')\delta_{ll'} \qquad \sum_{l=-\infty}^{+\infty}|\lambda l\rangle\langle\lambda' l| = \delta_{\lambda\lambda'}/2\pi. \tag{3.23}$$

The representations of the principal series $T_{\lambda,\varepsilon}$ with arbitrary non-zero ε can be constructed in terms of multivalued functions on a sheet of the hyperboloid ($\varepsilon = 0$ corresponds to the single-valued representations). The eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{L}^0$ are the same adjoint Legendre functions (3.22) with $l = n + \varepsilon$, n integer, and with scalar product (3.23), where the factor $\tanh(\pi\lambda/2)$ has to be replaced by one $\tanh(\pi(\lambda/2 + \mathrm{i}\varepsilon))$ [15]. At $\varepsilon = 1/2$ (double-valued representations) and $j = -1/2$, the representation is reducible and is split into two representations with the highest weight $l = -1/2$ and with the lowest weight $l = 1/2$, the corresponding functions have the form $(\sinh\theta)^{-1/2}\,\mathrm{e}^{\mp\mathrm{i}\phi/2}$, according to (3.20).

(2) $m = 0$. The operators $\hat{L}^{\mu}$ (3.15), and

$$\hat{L}_{\pm} = \mathrm{e}^{\pm\mathrm{i}l\phi}(p\partial/\partial p \pm \partial/\partial\phi) \qquad \hat{\boldsymbol{L}}^2 = p\partial/\partial p(p\partial/\partial p + 1) \tag{3.24}$$

act in the space of functions on the cone $\boldsymbol{p}^2 = 0$. One can remark that the expression (3.24) for $\hat{L}_{\pm}$ passes into the expression (A20) on the complex cone (A9) after the replacement of p by ρ^2. The scalar products on these manifolds differ only by the limits of integration over the angle ϕ ($[-2\pi, 2\pi]$ or $[0, 2\pi]$). Thus, the representations of the principal series $T_{\lambda,\varepsilon}$ can be constructed in the space of functions on the cone, however, only the representations with $\varepsilon = 0$ are single-valued and the representation with $\varepsilon = 1/2$ are double-valued.

According to (A21) the wavefunction of a massless particle with the fixed $j = -1/2 + \mathrm{i}\lambda/2$ and with the projection l has the form in the momentum representation

$$|\lambda l\rangle = p^{-1/2+\mathrm{i}\lambda/2}\,\mathrm{e}^{\mathrm{i}l\phi}/2\pi \qquad \langle\lambda l \mid \lambda' l'\rangle = \delta((\lambda-\lambda')/2)\delta_{ll'}. \tag{3.25}$$

(3) $m \neq 0$ and is imaginary. The operators $\hat{L}^\mu$ (3.18) and

$$\hat{L}_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}(\pm\mathrm{i}\tanh\theta\,\partial/\partial\phi + \partial/\partial\theta) \qquad \hat{\boldsymbol{L}}^2 = \frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} - \frac{1}{\cosh^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{3.26}$$

act in the space of functions on one-sheeted hyperboloids. Unitary IRs of the discrete series can be realized in such a space. The result of the action of the arising operator $\hat{L}_+$ on the highest weights $f_j(\theta)\,\mathrm{e}^{\mathrm{i}j\phi}$ of the discrete negative series IRs must be zero,

$$\hat{L}_+ f_j(\theta)\,\mathrm{e}^{\mathrm{i}j\phi} = \mathrm{e}^{\mathrm{i}(j+1)\phi}(-j\tanh\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0$$

thus, $f_j(\theta) = (\cosh\theta)^j$. By analogy, we get the expression $(\cosh\theta)^j\,\mathrm{e}^{-\mathrm{i}j\phi}$ for the lowest weight of the discrete positive series. Normalizing these functions by means of the scalar product (3.17) and denoting them as $Y_{jj}(\theta,\phi)$ and $Y_{j-j}(\theta,\phi)$, we can write

$$Y_{j\pm j}(\theta,\phi) = \left(\frac{(-2j-2)!!}{\pi^2(-2j-3)!!}\right)^{1/2}(\cosh\theta)^j\,\mathrm{e}^{\pm\mathrm{i}j\phi}. \tag{3.27}$$

The functions $Y_{j\,l}(\theta,\phi), l < j$ (IR T_j^-) can be derived by the action of the lowering operator $\hat{L}_-$ on the highest weight $Y_{j-j}(\theta,\phi)$, and the functions $Y_{j\,l}(\theta,\phi)$, $l > -j$ (IR T_j^+) can be derived by the action of the arising operator $\hat{L}_+$ on the lowest weight $Y_{j\,j}(\theta,\phi)$. By analogy with the spherical functions we will call (3.27) the functions of the one-sheeted hyperboloid. The wavefunctions of tachyons in $2+1$ dimensions have the form,

$$|jl\rangle = Y_{j\,l}(\theta,\phi) \qquad \langle\lambda l \mid \lambda' l'\rangle = \delta_{\lambda\lambda'}\delta_{ll'} \tag{3.28}$$

where $j \leqslant -1$ and is integer (for the multivalued IR $j < -1/2$, and non-integer), whereas the momentum projection $l \geqslant |j|$. The functions (3.28), similar to the ordinary spherical functions, differ from the adjoint Legendre functions P_l^j by a factor only.

In the general case one has to consider eigenfunctions of the operators $\hat{\boldsymbol{L}}^2$ and $\hat{L}^0$ with the eigenvalues $j(j+1)$ and l. These functions have the form $f(\theta)\,\mathrm{e}^{\mathrm{i}l\phi}$, where $f(\theta)$ obeys the equation

$$\left(\frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} + \frac{1}{\cosh^2\theta}l^2\right)f(\theta) = j(j+1)f(\theta) \tag{3.29}$$

which coincides with one for the adjoint Legendre functions,

$$\left((1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} - \frac{l^2}{(1-z^2)}\right)P_j^l(z) = -j(j+1)P_j^l(z)$$

at $z = \mathrm{i}\sinh\theta$. At $j \leqslant -1$ we get the above considered IR of the discrete series. The functions $P_j^l(\mathrm{i}\sinh\theta)$ at $j = -1/2 + \mathrm{i}\lambda/2$ could correspond to the principal series of the unitary IRs, but the corresponding norm is divergent in this case.

Thus, our consideration shows: in the course of the reduction on the subgroup $SO(2,1)$ that the representations $T_m^\pm(g)$ and $T_0^\pm(g)$ of $\tilde{M}(2,1)$ with real (in particular zero) mass are split into IRs of the principal series, $j = -1/2 + \mathrm{i}\lambda, \hat{\boldsymbol{L}}^2 \leqslant -1/4$, whereas l are arbitrary integer. For tachyons, the representations $T_m(g)$ are split into IR of the discrete series, $j \leqslant -1$ and integer, $\hat{\boldsymbol{L}}^2 = j(j+1) \geqslant 0$ (i.e. the space component of the angular momentum L^0 is greater than the bust ones). For the tachyons the absolute value of the projection l cannot be less than $|j|$, in particular, l cannot be zero.

Below we present three sets of wavefunctions of scalar particles, which are eigenfunctions for the commuting operators, $\{\hat{p}_\mu\}$, $\{\hat{\boldsymbol{p}}^2, \hat{p}_0, \hat{L}^0\}$ and $\{\hat{\boldsymbol{p}}^2, \hat{\boldsymbol{L}}^2, \hat{L}^0\}$ respectively.

(1) States with a given momentum, $f(x) = \mathrm{e}^{-\mathrm{i}px}$.

(2) States with a given energy p_0 and angular momentum projection l (in x-representation),

$$f(x) = \mathrm{e}^{\mathrm{i}p_0 x^0 + \mathrm{i}l\phi} J_l(\rho\sqrt{p_0^2 - m^2}) \tag{3.30}$$

where ρ, ϕ are the polar coordinates in the x^1, x^2 plane, and J_l are Bessel functions.

(3) States (3.21) in the p-representation with a given orbital momentum j and its projection l. According to (3.22), (3.25) and (3.28), we have three cases:

$$m > 0 \qquad |\lambda l\rangle = \tilde{P}_j^l(\cosh\theta)\,\mathrm{e}^{\mathrm{i}l\phi} \qquad j = -1/2 + \mathrm{i}\lambda/2 \tag{3.31}$$

where θ and ϕ are coordinates on two sheet hyperboloids $p^2 = m^2 > 0$, and $\tilde{P}_j^l$ are adjoint Legendre functions;

$$m = 0 \qquad |\lambda l\rangle = p^{-1/2+\mathrm{i}\lambda/2}\,\mathrm{e}^{\mathrm{i}l\phi} \tag{3.32}$$

where θ and ϕ are coordinates on the light cone $p^2 = 0$;

$$m\text{—imaginary} \qquad |jl\rangle = Y_{jl}(\theta, \phi) \tag{3.33}$$

where θ and ϕ are coordinates on one sheet hyperboloids $p^2 = m^2 < 0$, and $Y_{jl}(\theta, \phi)$ are one sheet hyperboloid functions (3.28).

4. Generalized regular representation and 2 + 1 spin

In the previous section we considered the quasi-regular representation, which produces a description of scalar fields or spinless particles. To get a complete picture of all possible representations one has to turn to the so-called generalized regular representation (GRR) [13–15]. The GRR acts in the space of functions $f(g)$ on the group. The left GRR $T_L(g)$ and the right GRR $T_R(g)$ are defined as

$$T_L(g) f(g_0) = f(g^{-1}g_0) \tag{4.1}$$

$$T_R(g) f(g_0) = f(g_0 g). \tag{4.2}$$

It is known that any IR of a group is equivalent to that of a sub-representation of the left (right) GRR [13]. Taking this into account, we can construct a GRR of $\tilde{M}(2, 1)$ in the parametrization (2.5)–(2.7), where $g_0 \leftrightarrow (x, z) \leftrightarrow (X, Z)$, $g \leftrightarrow (x, z) \leftrightarrow (A, U)$,

$$X = \begin{pmatrix} x^0 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 \end{pmatrix} \qquad Z = \begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

$$A = \begin{pmatrix} a^0 & a^1 - \mathrm{i}a^2 \\ a^1 + \mathrm{i}a^2 & a^0 \end{pmatrix} \qquad U = \begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix}. \tag{4.3}$$

Using the composition law (2.8), one can get

$$T_L(g) f(x, z) = f(g^{-1}x, g^{-1}z) \qquad g^{-1}x \leftrightarrow U^{-1}(X - A)(U^{-1})^\dagger \qquad g^{-1}z \leftrightarrow U^{-1}Z \tag{4.4}$$

$$T_R(g) f(x, z) = f(xg, zg) \qquad xg \leftrightarrow X + ZAZ^\dagger \qquad zg \leftrightarrow ZU. \tag{4.5}$$

According to (4.4), X is transformed with respect to the adjoint (vector) representation and Z with respect to the spinor representation of $SU(1, 1)$. One can also see that Z is invariant under translations. If one is restricted to Z-independent functions (i.e. by the functions on the coset space $\tilde{M}(2,1)/SU(1,1)$), then (4.4) reduces to the quasi-regular representation (3.1), which corresponds to the scalar field case. If one restricts itself to X-independent functions, then (4.4) and (4.5) reduce to the left and the right GRR of $SU(1,1)$.

Calculating generators, which correspond to the parameters a^μ and $-\alpha_\mu$, in the left GRR (4.4), we get

$$\hat{p}_\mu = \mathrm{i}\partial/\partial x^\mu \qquad \hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu \tag{4.6}$$

where $\hat{L}^\mu$ are the angular momentum operators (3.3), and $\hat{S}^\mu$ are spin operators,

$$\hat{S}^0 = -\frac{1}{2}V\sigma^3\partial_V + \frac{1}{2}\bar{V}\sigma^3\partial_{\bar{V}} \qquad \hat{S}^1 = \frac{\mathrm{i}}{2}V\sigma^2\partial_V - \frac{\mathrm{i}}{2}\bar{V}\sigma^2\partial_{\bar{V}}$$
$$\hat{S}^2 = \frac{\mathrm{i}}{2}V\sigma^1\partial_V + \frac{\mathrm{i}}{2}\bar{V}\sigma^1\partial_{\bar{V}} \qquad [\hat{S}^\mu, \hat{S}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{S}_\eta \qquad [\hat{S}^\mu, \hat{p}_\nu] = 0 \tag{4.7}$$

and $V = (z_1\bar{z}_2)$, $\bar{V} = (\bar{z}_1 z_2)$. The algebra of the generators (4.6) has the form

$$[\hat{p}_\mu, \hat{p}_\nu] = 0 \qquad [\hat{p}^\mu, \hat{J}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{p}_\eta \qquad [\hat{J}^\mu, \hat{J}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{J}_\eta. \tag{4.8}$$

We denote the generators of the right GRR by the same letters but they are underlined. The generators $\underline{\hat{J}}^\mu$ do not depend on x and are only expressed in terms of z,

$$\underline{\hat{p}}_\mu = -(\Lambda^{-1})^\nu_\mu \hat{p}_\nu \quad (\text{or } \underline{\hat{P}} = -Z^{-1}\hat{P}(Z^{-1})^\dagger) \qquad \underline{\hat{J}}^\mu = \underline{\hat{S}}^\mu \tag{4.9}$$

$$\underline{\hat{S}}^0 = \frac{1}{2}\chi\sigma^3\partial_\chi - \frac{1}{2}\bar{\chi}\sigma^3\partial_{\bar{\chi}} \qquad \underline{\hat{S}}^1 = \frac{\mathrm{i}}{2}\chi\sigma^2\partial_\chi - \frac{\mathrm{i}}{2}\bar{\chi}\sigma^2\partial_{\bar{\chi}}$$
$$\underline{\hat{S}}^2 = -\frac{\mathrm{i}}{2}\chi\sigma^1\partial_\chi - \frac{\mathrm{i}}{2}\bar{\chi}\sigma^1\partial_{\bar{\chi}} \tag{4.10}$$

where $\chi = (z_1 z_2)$, $\bar{\chi} = (\bar{z}_1\bar{z}_2)$. All the right generators commute with all the left generators and obey the same commutation relations (4.8). The operator $\hat{\boldsymbol{p}}^2 = \underline{\hat{\boldsymbol{p}}}^2$ and Pauli–Lubanski scalar $\hat{W} = \hat{p}\hat{J} = \underline{\hat{p}}\underline{\hat{J}}$ are the Casimir operators. Thus, IRs of $\tilde{M}(2,1)$ can be marked by their eigenvalues.

It follows from (3.3) that $\hat{p}\hat{L} = 0$, so that always $\hat{W} = \hat{p}\hat{S}$. The operator $\hat{W}$ commutes with the total angular momentum operators $\hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu$, but not with the orbital momentum operators $\hat{L}^\mu$ and spin operators $\hat{S}^\mu$ separately. The operator of spin square $\hat{\boldsymbol{S}}^2 = \underline{\hat{\boldsymbol{J}}}^2$ commutes with all the generators of the left GRR. That means that objects, which are transformed under the left GRR or under its sub-representations, can also be marked by eigenvalues of this operator. However, that operator does not commute with the generators $\underline{\hat{p}}_\mu$ of the right GRR, $[\underline{\hat{p}}^\mu, \underline{\hat{\boldsymbol{J}}}^2] = \mathrm{i}\epsilon^{\mu\nu\eta}(\underline{\hat{p}}_\nu\underline{\hat{J}}_\eta + \underline{\hat{J}}_\eta\underline{\hat{p}}_\nu)$, similar to the left GRR case, $[\hat{p}^\mu, \hat{\boldsymbol{J}}^2] = \mathrm{i}\epsilon^{\mu\nu\eta}(\hat{p}_\nu\hat{J}_\eta + \hat{J}_\eta\hat{p}_\nu)$. Thus, the square of spin is not a conserved quantity in all the right representations, but $\hat{\boldsymbol{J}}^2$ is.

Making the Fourier transformation (3.6) in the variables x, i.e. considering representations in the space of functions $\varphi(p,z)$, one can get an analogue of the formulae (4.4) and (4.5) in this representation,

$$T_L(g)\varphi(p,z) = \mathrm{e}^{\mathrm{i}ap'}\varphi(p', g^{-1}z) \qquad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^\dagger \tag{4.11}$$
$$T_R(g)\varphi(p,z) = \mathrm{e}^{-\mathrm{i}a'p}\varphi(p, zg) \qquad a' \leftrightarrow A' = ZAZ^\dagger \tag{4.12}$$

where P is defined by (3.8). It can be seen that the combination of $|z_1|^2 - |z_2|^2$ and p^2 is conserved under the transformations (4.11) and (4.12). The former is always equal to one and the latter to m^2, and depends on the representation. Z and P are defined by six real parameters. Three of them (namely, $\underline{P} = -Z^{-1}P(Z^{-1})^\dagger$ for the left GRR or P for the right GRR) are fixed and only three of them vary under the group transformations (for the left GRR two of them set the direction of the momentum).

The classification of the orbits with respect to the eigenvalues of the operator $\hat{p}^2$ is completely similar to that in section 3 for the spinless case. These are orbits $O_m^\pm$ for real $m \neq 0$, $O_0^\pm$ and O_0^0 for $m = 0$, and finally O_m for imaginary m. However, to describe the IR only one parameter m is not enough, one needs to know the characteristics connected with the spin.

We note that the left and the right GRR are equivalent, $\hat{C}T_R(g) = T_L(g)\hat{C}$, where $\hat{C}f(g_0) = f(g_0^{-1})$. Because of that, and also since the left representations are more adequate to describe physical fields, we are going to consider in more detail only the left GRR of $\tilde{M}(2,1)$.

Consider the left GRR, which acts in the space of functions $f(x,z)$, $f'(x,z) = T_L(g)f(x,z) = f(g^{-1}x,\ g^{-1}z)$. It is easy to remark that

$$f'(x',z') = f(x,z) \tag{4.13}$$

where

$$x' = gx = \Lambda x + a \leftrightarrow U(X+A)U^\dagger \qquad z' = gz \leftrightarrow UZ. \tag{4.14}$$

Thus one can reduce the problem of the classification of left representations to that of the scalar functions (4.13)–(4.14), using the general scheme of the harmonic analysis [11, 13].

To classify the functions $f(x,z)$ we are going to use besides the Casimir operators $\hat{\boldsymbol{p}}^2$, $\hat{W}$, the operator of the spin square $\hat{\boldsymbol{S}}^2$, which commutes with all the generators of the left GRR. By means of this operator it is convenient to select IRs from the set of equivalent ones, and, moreover, to classify IRs in the special case of zero eigenvalues of the Casimir operators, where the functions (4.13) do not depend on x. In the latter case IRs of the Poincare group coincide, in fact, with those of the Lorentz group.

Let us consider in this connection the discrete basis $\overline{R}_{S\zeta}(z)$ of the Lorentz group representation $T_S(g)$,

$$\begin{aligned} &\hat{\boldsymbol{S}}^2\overline{R}_{S\zeta}(z) = S(S+1)\overline{R}_{S\zeta}(z) \qquad \hat{S}^0\overline{R}_{S\zeta}(z) = \zeta\overline{R}_{S\zeta}(z) \\ &\overline{R}'_S(z) = T_S(g)\overline{R}_S(z) = \overline{R}_S(g^{-1}z) \end{aligned} \tag{4.15}$$

where $\overline{R}_S(z)$ is a column with the components $\overline{R}_{S\zeta}(z)$. The number S marks the IR of the Lorentz group and further we will call S the Lorentz spin. The possible values of S and the corresponding spectrum of ζ depends on the type of the Lorentz group representation, see the appendix and table A1. The eigenvectors $f(x,z)$ of the operator $\hat{\boldsymbol{S}}^2$ can be presented in the form

$$f(x,z) = \sum_\zeta \overline{\psi}_\zeta(x)\overline{R}_{S\zeta}(z) = \overline{\psi}(x)\overline{R}_S(z) \tag{4.16}$$

where $\overline{\psi}(x)$ is a line with components $\overline{\psi}_\zeta(x)$. On the other hand one can introduce a basis $R_{S\zeta}(z)$ of the contragradient [11] to the $T_S(g)$ representation. In terms of this basis a function $f(x,z)$ can be presented by the decomposition

$$f(x,z) = \sum_\zeta \psi_\zeta(x)R_{S\zeta}(z) = \psi(x)R_S(z) \qquad R'_S(z) = R'_S(z)T_S(g^{-1}) \tag{4.17}$$

where $R_S(z)$ is a line with the components $R_{S\zeta}(z)$ and $\psi_\zeta(x)$ is a column with the components $\psi_\zeta(x)$. In the case when the representation $T_S(g)$ and its contragradient are equivalent, which is the value for example for finite-dimensional IRs of the Lorentz group, one and the same function has both representations (4.16) and (4.17). Using (4.16) and (4.17), one can find

$$\overline{\psi}'(x') = \overline{\psi}(x)T_S(g) \qquad \psi'(x') = T_S(g^{-1})\psi(x).$$

The product $\overline{\psi}(x)\psi(x)$ is Poincaré invariant.

Thus the eigenvectors of $\hat{\boldsymbol{S}}^2$ can be described by the columns $\psi(x)$ (lines $\overline{\psi}(x)$) with the components $\psi_\zeta(x)$ ($\overline{\psi}_\zeta(x)$). Their dimensionality depends on the representation of the Lorentz group. Further we will call $\psi(x)$ the wavefunction in S-representation or simply the wavefunction. In such a form all the spinning operators can be realized as discrete matrices. Their explicit form can be easily found.

As is demonstrated in the appendix any IR of the Lorentz group can be constructed on the elements of the first column of the matrix Z (4.4). Thus one can be restricted by the functions $f(x,z)$, with $z = \{z_1, \bar{z}_2\}$ only. In this case eigenvectors of the operator $\hat{\boldsymbol{S}}^2$ are homogeneous functions in the variables z_1 and $\bar{z}_2$ of the power $2S$, and the discrete basis can be chosen in the form

$$R_{S\zeta}(z) = N_{S\zeta} z_1^{S-\zeta} \bar{z}_2^{S+\zeta}. \tag{4.18}$$

The Lorentz IR with $2S$ integer and positive are non-unitary and finite-dimensional, whereas unitary infinite-dimensional IRs correspond to $S < 0$ (discrete and supplementary series) and $S = -1/2 + \mathrm{i}\lambda/2$ (principal series).

Let $2S$ be integer and positive. (The case $S = 0$ corresponds to the scalar functions (3.1), which do not depend on z.) First consider $S = 1/2$. In this case the decomposition (4.17) can be written in the form

$$f(x,z) = \overline{\psi}_{-1/2}(x)z_1 + \overline{\psi}_{1/2}(x)\bar{z}_2 \qquad \hat{\boldsymbol{S}}^2 f = \tfrac{3}{4} f. \tag{4.19}$$

Applying the transformation (4.4) to this function

$$f'(x,z) = (\overline{\psi}'_{-1/2}(x)\overline{\psi}'_{1/2}(x)) \begin{pmatrix} z_1 \\ \bar{z}_2 \end{pmatrix} = (\overline{\psi}_{-1/2}(g^{-1}x)\overline{\psi}_{1/2}(g^{-1}x))U^{-1} \begin{pmatrix} z_1 \\ \bar{z}_2 \end{pmatrix}$$

we conclude that the line $\overline{\psi}(x) = (\overline{\psi}_{-1/2}(x)\overline{\psi}_{1/2}(x))$ is transformed under the spinor representation of the Lorentz group,

$$\overline{\psi}'(x') = \overline{\psi}(x)U^{-1}.$$

Taking into account the relation $U^{-1} = \sigma^3 U^\dagger \sigma^3$, which is valued for the $SU(1,1)$ matrices, we get the transformation low for the columns $\psi(x) = (\psi_{1/2}(x)\psi_{-1/2}(x))^T = \sigma^3 \overline{\psi}^\dagger$,

$$\psi'(x') = U\psi(x).$$

One can find that the same spinor ψ appears from the decomposition

$$f(x,z) = \psi_{1/2}(x)\bar{z}_2 - \psi_{-1/2}(x)z_1 = (\bar{z}_2 \ -z_1) \begin{pmatrix} \psi_{1/2}(x) \\ \psi_{-1/2}(x) \end{pmatrix} \qquad \hat{\boldsymbol{S}}^2 f = \frac{3}{4} f. \tag{4.20}$$

Thus, in the case under consideration, we have two equivalent descriptions. One in terms of functions (4.13)) and another in terms of lines $\overline{\psi}(x)$ or columns $\psi(x)$. One can find the action of the operators $\hat{S}^\mu$ in the latter representation,

$$\hat{S}^\mu \psi(x) = \tfrac{1}{2}\gamma^\mu \psi(x)$$

where

$$\gamma^\mu = (\sigma^3, \mathrm{i}\sigma^2, -\mathrm{i}\sigma^1) \qquad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu} \qquad [\gamma^\mu, \gamma^\nu] = -2\mathrm{i}\epsilon^{\mu\nu\lambda}\gamma_\lambda \tag{4.21}$$

are 2×2 γ-matrices in $2+1$ dimensions†. The functions $\psi = (\psi_{1/2}0)^T$ and $\psi = (0\psi_{-1/2})^T$ are eigenvectors for the operator $\hat{S}^0$ with the eigenvalues $(\pm 1/2)$.

The product $\overline{\psi}(x)\psi(x) = \overline{\psi}'(x')\psi'(x')$ is the scalar density, which is not positive defined.

The polynomials of the power $2S$ can be written in the form

$$f(x,z) = \sum_{n=0}^{2S} \overline{\psi}_{n-S}(x)(C_{2S}^n)^{1/2} z_1{}^{2S-n}\bar{z}_2^n = \overline{\psi}(x)\overline{R}_S(z) \tag{4.22}$$

where $\overline{\psi}(x)$ is a $(2S+1)$-component line, $\overline{R}_S(z)$ is a column with elements $(C_{2S}^n)^{1/2} z_1{}^{2S-n}\bar{z}_2^n$, $n = 0, 1, \ldots, 2S$, which is transformed with respect to the finite-dimensional IR $T_S(g^{-1})$ of the Lorentz group, $\overline{R}'_S(z) = T_S(g^{-1})\overline{R}_S(z)$, or in the form

$$f(x,z) = \sum_{n=0}^{2S} \psi_{S-n}(x)(C_{2S}^n)^{1/2}(-z_1)^n \bar{z}_2^{2S-n} = R_S(z)\psi(x) \qquad \hat{\boldsymbol{S}}^2 f = S(S+1)f \tag{4.23}$$

where $\psi(x)$ is $(2S+1)$-component column, $\psi(x) = \Gamma\overline{\psi}^{\dagger}(x)$, and $(\Gamma)_{nn'} = (-1)^n\delta_{nn'}$.

By analogy with the case $S = 1/2$ one finds

$$\overline{\psi}'(x') = \overline{\psi}(x)T_S(g) \qquad \psi'(x') = T_S(g^{-1})\psi(x). \tag{4.24}$$

Here the scalar density has the form $\overline{\psi}(x)\psi(x) = \psi^{\dagger}(x)\Gamma\psi(x)$. The operators $\hat{S}^\mu$ are $(2S+1)\times(2S+1)$ spin matrices S^μ in the space of columns $\psi(x)$, and are generators of $SU(1,1)$ in the representation T_S,

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(S-n) \qquad n = 0, 1, \ldots, 2S \\
(S^1)_{nn'} &= -\frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} - \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right) \\
(S^2)_{nn'} &= -\frac{\mathrm{i}}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} + \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right).
\end{aligned} \tag{4.25}$$

For the infinite-dimensional unitary IR of $SU(1,1)$ the values of S can be non-integer, $S < -1/2$ (discrete series), $-1/2 < S < 0$ (supplementary series), or complex, $S = -1/2+\mathrm{i}\lambda/2$ (principal series), see the appendix. Consider first representations with the highest or lowest weights. These are all representations of the discrete series $T_S^{\pm}$ and two representations of the principal series $T_{S,\varepsilon}$, which correspond to $S = -1/2$ and $\varepsilon = 1/2$, i.e. to half-integer spin projections. The eigenfunctions of the operator $\hat{\boldsymbol{S}}^2$ in the representations $T_S^{\pm}$ are negative power S quasi-polynomials (see (A15)),

$$\begin{aligned}
f^+(x,z) &= \sum_{n=0}^{\infty} \psi_n^+(x)(C_{2S}^n)^{1/2}(-z_1)^{2S-n}\bar{z}_2^n \\
f^-(x,z) &= \sum_{n=0}^{\infty} \psi_n^-(x)(C_{2S}^n)^{1/2}(-z_1)^n\ \bar{z}_2^{2S-n} \\
\psi^{\pm\prime}(x') &= T_S^{\pm}(g^{-1})\psi^{\pm}(x) \qquad C_{2S}^n = \left(\frac{(-1)^n\Gamma(n-2S)}{n!\Gamma(-2S)}\right)^{1/2}.
\end{aligned} \tag{4.26}$$

The representations of the positive and negative series are conjugated,

$$(T_S^+(g))^\dagger = T_S^-(g) \qquad (\psi^{\pm\prime}(x'))^\dagger = (\psi^{\pm}(x))^\dagger T_S^{\mp}(g).$$

† The generators (4.6) of the left GRR correspond to the parameters a^μ and $-\alpha_\mu$. If we take the generators which correspond to the parameters a^μ and α_μ then another non-equivalent representation for γ-matrices appears, which differs from (4.21) by a sign.

In contrast to the case of the finite-dimensional representations, here the scalar density is positively defined,

$$(\psi^+(x))^\dagger \psi^+(x) = \sum_{n=0}^{\infty} |\psi^+_{-S+n}(x)|^2 \qquad (\psi^-(x))^\dagger \psi^-(x) = \sum_{n=0}^{\infty} |\psi^-_{S-n}(x)|^2.$$

The possible eigenvalues ζ of the operator $\hat{S}^0$ obey the inequality $|\zeta| \geqslant |S| > 1/2$ for the IR of the discrete series. The spin projection ζ can take on only positive values for the representations T_S^+, $\zeta = -S + n$, and negative values for T_S^-, $\zeta = S - n$.

For the representations T_S^+ the spin matrices S^μ are

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(-S+n) \qquad n = 0, 1, 2, \ldots \\
(S^1)_{nn'} &= -\frac{\mathrm{i}}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} - \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right) \\
(S^2)_{nn'} &= \frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} + \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right).
\end{aligned} \tag{4.27}$$

For T_S^- representations S^1 is the same and S^0, S^2 change sign only.

In the case of unitary representations of the principal series, $S = -1/2 + \mathrm{i}\lambda/2$, the functions $f(x, z)$ are presented by the infinite sum,

$$f(x, z) = \sum_{n=-\infty}^{+\infty} \psi_{\varepsilon+n}(x)\, \mathrm{i}^n (-z_1)^{-1/2-\mathrm{i}\lambda/2-(\varepsilon+n)} \bar{z}_2^{-1/2-\mathrm{i}\lambda/2+(\varepsilon+n)} \qquad \hat{\boldsymbol{S}}^2 f = -\frac{1}{4}(1+\lambda^2) f. \tag{4.28}$$

The spin projection ζ can take on the values $\varepsilon + n$, where $\varepsilon \in [-1/2, 1/2]$, $n = 0, \pm 1, \ldots$. In the space of infinite-dimensional columns ψ with the elements $\psi_{\varepsilon+n}(x)$ the operators $\hat{S}^\mu$ have the form of corresponding infinite-dimensional matrices S^μ,

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(\varepsilon+n) \qquad n = 0, \pm 1, \pm 2, \ldots \\
(S^1)_{nn'} &= -\frac{\mathrm{i}}{2}(\delta_{n\ n'+1}(-1/2+\varepsilon+n-\mathrm{i}\lambda/2) - \delta_{n+1\ n'}(1/2+\varepsilon+n+\mathrm{i}\lambda/2)) \\
(S^2)_{nn'} &= \frac{1}{2}(\delta_{nn'+1}(-1/2+\varepsilon+n-\mathrm{i}\lambda/2) + \delta_{n+1n'}(1/2+\varepsilon+n+\mathrm{i}\lambda/2)).
\end{aligned} \tag{4.29}$$

As a result of the unitarity of the representations under consideration, the corresponding scalar density

$$\psi^\dagger(x)\psi(x) = \sum_{n=-\infty}^{\infty} |\psi_{\varepsilon+n}(x)|^2$$

is positively defined.

In the case of the unitary infinite-dimensional representations of the principal and discrete series the matrices S^1 and S^2 are Hermitian, whereas in the case of the finite-dimensional non-unitary representations aready considered they are anti-Hermitian. In the space of columns with elements ψ_ζ the matrices S^1 and S^2 have non-zero elements only on the secondary diagonals.

The spin projection ζ can take on non-integer values for some IRs of the principal and discrete series. These IRs can be used to describe the anions [4].

5. Relativistic wave equations and IRs of $\tilde{M}(2, 1)$

5.1. Relativistic wave equations

As is known, wavefunctions of relativistic particles are identified with vectors of IR spaces of the corresponding Poincare group. Thus the problem of the construction of the relativistic wave equations for particles with different spins can be solved by means of a decomposition of the left GRR of the $\tilde{M}(2, 1)$ group.

Consider functions $f(x, z)$, which are transformed under the left GRR of $\tilde{M}(2, 1)$, and which are eigenvectors for the Casimir operators $\hat{\boldsymbol{p}}^2$, $\hat{W} = \hat{p}\hat{S}$, and for the operator $\hat{\boldsymbol{S}}^2$, which commute with all the generators of the left GRR,

$$(\hat{\boldsymbol{p}}^2 - m^2)f(x, z) = 0 \tag{5.1}$$

$$(\hat{p}_\mu \hat{S}^\mu - K)f(x, z) = 0 \tag{5.2}$$

$$(\hat{\boldsymbol{S}}^2 - S(S + 1))f(x, z) = 0. \tag{5.3}$$

The equations (5.1)–(5.3) define some sub-representation of the left GRR of $M(2, 1)$, which is characterized by mass m, Lorentz spin S, and by the eigenvalue K of Lubanski–Pauli operator. Possible values of K can be easily described in the massive case. Here we can use a rest frame, where $\hat{p}_\mu \hat{S}^\mu = \hat{S}^0 m \operatorname{sign} p_0$. Thus, for particles $K = sm$ and for antiparticles $K = -sm$, where the spectrum s coincides with one of the operators $\hat{S}^0$. The latter spectrum depends on the representation of the Lorentz group, see the appendix and the table A1. At $m = 0$ we suppose $K = 0$, that is true for IRs with a finite number of spinning degrees of freedom. The general cases $m = 0$ and m imaginary will be discussed below.

At S fixed and in the S-representation the equations (5.1)–(5.2) have the form

$$(\hat{p}^2 - m^2)\psi(x) = 0 \tag{5.4}$$

$$(\hat{p}_\mu S^\mu - sm)\psi(x) = 0 \tag{5.5}$$

where $\psi(x)$ are columns and S^μ are matrices, described in the previous section. They obey the commutation relations of the $SU(1, 1)$ group,

$$[S^\mu, S^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta} S_\eta.$$

Let us describe possible cases, which correspond to finite-dimensional non-unitary IRs, and to infinite-dimensional unitary IRs of the latter group.

(1) Consider finite-dimensional and non-unitary IRs of $SU(1, 1)$. In this case S has to be positive, integer or half-integer. According to (5.5),

$$\psi^\dagger(x)(\mathrm{i}S^{\dagger\mu}\overleftarrow{\partial_\mu} + sm) = 0.$$

It follows from the explicit expressions for S^μ (4.22) that $S^{\dagger\mu} = \Gamma S^\mu \Gamma$, where $(\Gamma)_{nn'} = (-1)^n \delta_{nn'}$. The function $\overline{\psi} = \psi^\dagger\Gamma$ obeys the equation

$$\overline{\psi}(x)(\mathrm{i}S^\mu \overleftarrow{\partial_\mu} + sm) = 0. \tag{5.6}$$

As a consequence of (5.5) and (5.6), the continuity equation holds

$$\partial_\mu j^\mu = 0 \qquad j^\mu = \overline{\psi} S^\mu \psi. \tag{5.7}$$

At $S = 1/2$ the density $j^0 = \overline{\psi} S^0 \psi$ is positively defined (the scalar density $\overline{\psi}\psi$ is not positively defined, as was mentioned before).

At $S = 1/2$ the equation (5.5) can be rewritten in the form of a $2+1$ Dirac equation,

$$(\hat{p}_\mu \gamma^\mu - m)\psi(x) = 0 \tag{5.8}$$

where $\gamma^\mu = 2S^\mu$ are γ-matrices in $2+1$ dimensions (4.21).

Let us consider the states $f(x, z) = \mathrm{e}^{-\mathrm{i}px}(Az_1 + Bz_2)$ with a definite momentum. The combination $|A|^2 - |B|^2 = C$ remains constant under the $\tilde{M}(2, 1)$ transformations. One can set A or B to be zero in a certain reference frame, depending on the sign of C. In the rest frame we get two wavefunctions, which cannot be connected by any $\tilde{M}(2, 1)$ transformation, $\mathrm{e}^{-\mathrm{i}p_0x^0}z_1$ $(C > 0)$, $\mathrm{e}^{-\mathrm{i}p_0x^0}z_2$ $(C < 0)$. They correspond to two different directions of the spin projection on the axis x^0. Representations of $\tilde{M}(2, 1)$ at $m > 0$ and $S = 1/2$ are split into two IR, which correspond to particles with spin projections $s = 1/2$ and $s = -1/2$.

The case $C = 0$, $f(x, z) = A\,\mathrm{e}^{-\mathrm{i}p^0x^0}(\mathrm{e}^{\mathrm{i}\phi_1}z_1 + \mathrm{e}^{\mathrm{i}\phi_2}z_2)$, $A \neq 0$, corresponds to the massless particle. Indeed, a straightforward calculation shows that the action of the operator $\hat{p}\hat{S}$ on the function $(\mathrm{e}^{\mathrm{i}\phi_1}z_1 + \mathrm{e}^{\mathrm{i}\phi_2}z_2)$ gives zero at $\hat{p}^0 = p$, $\hat{p}^1 = p\cos\varphi$, $\hat{p}^2 = p\sin\varphi$, $\varphi = \varphi_1 - \varphi_2$ (see also (5.35)). Thus at $S = 1/2$ we have three cases in accordance with possible values of the Casimir operator $\hat{p}\hat{S}$ $(\pm m/2, 0)$.

At $S = 1$ the decomposition (4.17) has the following form

$$f(x, z) = \psi_1(x)\bar{z}_2^2 - \psi_0(x)\sqrt{2}z_1\bar{z}_2 + \psi_{-1}(x)z_1^2 \tag{5.9}$$

where $\psi(x) = (\psi_1(x)\psi_0(x)\psi_{-1}(x))^T$ is subjected to the equation (5.5)

$$(\hat{p}_\mu S^\mu - sm)\psi(x) = 0 \tag{5.10}$$

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad S^1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^2 = -\frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the spin projection s takes on the values $\pm 1, 0$. If one introduces the new (Cartesian) components $\mathcal{F}_\mu$, $\mathcal{F}_1 = -(\psi_{-1} + \psi_1)/\sqrt{2}$, $\mathcal{F}_\in = -\mathrm{i}(\psi_1 - \psi_{-1})/\sqrt{2}$, $\mathcal{F}_0 = \psi_0$, instead of the components $\psi_1(x)$, $\psi_0(x)$, $\psi_{-1}(x)$ (cyclic components), then (5.5) takes the form

$$\partial_\mu \varepsilon^{\mu\nu\eta}\mathcal{F}_\eta + sm\mathcal{F}^\nu = 0. \tag{5.11}$$

A transversality condition follows from (5.11),

$$\partial_\mu \mathcal{F}^\mu = 0. \tag{5.12}$$

One can see now that the equations (5.11) are in fact field equations of the so-called 'self-dual' free massive field theory [24], with the Lagrangian

$$\mathcal{L}_{\mathrm{SD}} = \frac{1}{2}\mathcal{F}_\mu\mathcal{F}^\mu - \frac{s}{2m}\varepsilon^{\mu\nu\lambda}\mathcal{F}_\mu\partial_\nu\mathcal{F}_\lambda = 0. \tag{5.13}$$

As remarked in [25] this theory is equivalent to the topologically massive gauge theory [1] with the Chern–Simons term. Indeed, the transversality condition (5.12) can be viewed as a Bianchi identity, which allows the introduction of gauge potentials A_μ, namely a transverse vector can be written (in topologically trivial spacetime) as a curl:

$$\mathcal{F}^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\nu = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}$$

where $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ is the field strength. Thus, $\mathcal{F}^\mu$ appears to be dual field strength, which is a tree-component vector in $2+1$ dimensions. Then (5.11) implies the following equations for $F_{\mu\nu}$

$$\partial_\mu F^{\mu\nu} + \frac{sm}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{5.14}$$

which are the field equations of the topologically massive gauge theory with the Lagrangian

$$\mathcal{L}_{CS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{sm}{4}\varepsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda. \tag{5.15}$$

One can find that finite transformations of $\tilde{M}(2,1)$ act on the Cartesian components as $\mathcal{F}'^{\nu}(x') = \Lambda^{\nu}_{\mu}\mathcal{F}^{\mu}(x)$. Here the combination $\bar{\mathcal{F}}_\mu\mathcal{F}^\mu = C(x)$ is preserved. C does not depend on x for states with a definite momentum. The case $C > 0$ corresponds to particles with real mass $m \neq 0$, the case $C = 0$ corresponds to massless particles. The correspondent wavefunctions will be presented below.

If a particle has integer or half-integer spin projection s, then the correspondent representation of $SU(1,1)$ of a minimal dimension is the finite-dimensional $T_S(g)$, where $S = |s|$, and $\dim T_S = 2S + 1$. To describe states with fractional spin projections one has to consider infinite-dimensional representations $SU(1,1)$.

(2) Consider now unitary infinite-dimensional IRs of $SU(1,1)$. In this case S can be non-integer, $S < -1/2$ (discrete series), $-1/2 < S < 0$ (supplementary series), or complex, $S = -1/2 + \mathrm{i}\lambda/2$ (principal series), see the appendix. Matrices S^μ are Hermitian and according to (5.5) the conjugated equation has the form

$$\psi^\dagger(x)(\mathrm{i}S^\mu \overleftarrow{\partial_\mu} + sm) = 0. \tag{5.16}$$

As a consequence of (5.5) and (5.16) the continuity equation holds

$$\partial_\mu j^\mu = 0 \qquad j^\mu = \psi^\dagger S^\mu \psi. \tag{5.17}$$

In IR of the discrete positive (negative) series $j^0 = \psi^\dagger S^0 \psi$ is positively (negatively) defined. Besides, for unitary IRs the scalar density $\psi^\dagger\psi$ is also positively defined in contrast to the finite-dimension case. For a discrete positive series s can take on only positive values, $s = -S + n$, and for negative ones only negative $s = S - n$, $n = 0, 1, 2, \ldots$. The case $s = \pm S$ has already been considered [4, 20, 21].

There are cases when the equations (5.4) and (5.5) are dependent. Indeed, multiplying (5.5) by $\hat{p}_\mu S^\mu + ms$ one gets

$$(\hat{p}_\mu S^\mu + ms)(\hat{p}_\mu S^\mu - ms)\psi(x) = (\hat{p}_\mu\hat{p}_\nu\{S^\mu, S^\nu\} - m^2s^2)\psi(x) = 0. \tag{5.18}$$

In the particular case $S = 1/2$ we have $s = \pm 1/2$, $S^\mu = \gamma^\mu/2$ and (5.18) is merely the Klein–Gordon equation (5.4). In the general case the matrices S^μ are not γ-matrices in higher dimensions and the squared equation (5.18) does not coincide with the Klein–Gordon equation.

As one can see from the consideration presented, the construction of the relativistic wave equations in $2 + 1$ dimensions is, in a sense, simpler than one in $3 + 1$ dimensions. That is connected with the vectorial nature of the operators of the angular momentum and of the spin. In $(3 + 1)$-dimensional case the above mentioned operators are tensors and it is namely this that complicates the problem.

Different IRs of $\tilde{M}(2,1)$ with $m \neq 0$ are marked by the spin projection s. However, one can see from the previous consideration, that the classification by the Lorentz spin S is also useful. S defines the dimension of matrix representation of the spin operators in (5.4) and (5.5).

One can easily see that massive particles have only one polarization state. Indeed, in the rest frame the equation (5.5) has the form

$$(S^0 - s)\psi = 0. \tag{5.19}$$

The spectrum s coincides with the spectrum of the operator S^0, which is not degenerated as was demonstrated above. Thus a fixation of s leads to only one solution of (5.5).

For $S = 1/2$ and $S = 1$ that property was demonstrated explicitly in [4]. One can reach the same conclusion, remarking that the non-relativistic group of movements is $M(2) = T(2)\times)SO(2)$, where the group $SO(2)$, which describes the spin, is an Abelian one and has only one-dimensional IRs.

In the case of the infinite-dimensional unitary representations of the $2+1$ Lorentz group, it is easier to deal with the functions $f(x, z)$, but not with an infinite number of their components $\psi_\zeta(x)$ in the S-representation.

As an example let us consider the plane wave solutions at $m > 0$. For $S = 1/2$ and $S = 1$ such solutions were analysed in [4] where it was noted that all the components are connected, that means that the number of spinning degrees of freedom is one. Here we are going to present similar consideration for all the representations of the $2+1$ Lorentz group, which have lowest weights, namely, for finite-dimensional T_S ($S > 0$, integer or half-integer), and for infinite-dimensional unitary representations T_S^+ ($S \leqslant -1/2$).

The wavefunction in the rest frame, which corresponds to the spin projection $s = -S$, has the form $z_1^{2S}\Psi(p_0)$, $p_0 = E = \pm m$. Acting on it by finite transformations, we get at $E > 0$ a solution in the form of the plane wave, which is characterized by the momentum p,

$$f(p, z) = (z_1\bar{u}_1 - \bar{z}_2 u_2)^{2S}\Psi(p) \qquad P = U^{-1}P_0(U^{-1})^\dagger \qquad P_0 = mI. \tag{5.20}$$

The momentum p does not depend on the parameter ϕ, $p^0 = E = m\cosh\theta$, $-p_1 + \mathrm{i}p_2 = m\sinh\theta\,\mathrm{e}^{\mathrm{i}\omega}$. Let us put $\phi = -\omega$ (in this case u_1 is real). Using the relations (2.7), one can express the parameters $\bar{u}_1$ and u_2 via the momentum p,

$$\begin{pmatrix} u_2 \\ \bar{u}_1 \end{pmatrix} = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} -p_1 + \mathrm{i}p_2 \\ E + m \end{pmatrix}. \tag{5.21}$$

In the case of finite-dimensional representations one can get $2S + 1$ components $\psi_\zeta(p)$ as coefficients in the decomposition of the function (5.20),

$$\psi(p) = \begin{pmatrix} \psi_S \\ \dots \\ \psi_{-S} \end{pmatrix} = \begin{pmatrix} u_2^{2S} \\ \dots \\ \bar{u}_1^{2S} \end{pmatrix} \Psi(p) \tag{5.22}$$

$$\psi_\zeta(p) = (C_{2S}^{S+\zeta})^{1/2}\bar{u}_1^{S-\zeta}u_2^{S+\zeta} = (C_{2S}^{S+\zeta})^{1/2}\frac{(E+m)^{S-\zeta}(-p_1 + \mathrm{i}p_2)^{S+\zeta}}{(2m(E+m))^S}\Psi(p). \tag{5.23}$$

In the particular case $S = 1/2$ we get [4],

$$\psi(p) = \frac{1}{\sqrt{2m(E-m)}} \begin{pmatrix} -p_2 + \mathrm{i}p_1 \\ E + m \end{pmatrix} \Psi(p). \tag{5.24}$$

For representations of discrete and principal series similar results hold. For example, in the former case one can get the formula (5.23), where C_{2S}^n are the coefficients from (4.26) and $\zeta = -S, -S+1, \dots$.

Among the above considered relativistic wave equations are ones which describe particles with fractional real spin. These equations are connected with unitary multivalued IRs of the Lorentz group and can be used to describe anyons. In spite of the fact that the number of independent polarization states for massive $2+1$ particles is one, the vectors of the corresponding representation space have an infinite number of components in S-representation. Thus, z-representation is more convenient in this case.

5.2. Dirac equation and CS evolution

It turns out that the $2+1$ Dirac equation appears also in the case of the infinite-dimensional unitary IRs of the $2+1$ Lorentz group (discrete and principal series with highest or lowest weights) as an equation for CS evolution. To see that, let us take, for example, spinning CS, related to the highest (lowest) weight of the IR T_S^- (T_S^+) (see the appendix),

$$\psi_u^-(x,z) = (z_1\bar{u}^2(x) + \bar{z}_2\bar{u}^1(x))^{2S} \tag{5.25}$$

$$\psi_u^+(x,z) = (z_1u^1(x) + \bar{z}_2u^2(x))^{2S} \qquad |u^1|^2 - |u^2|^2 = 1. \tag{5.26}$$

Here S can take on the value $-1/2$, that corresponds to the principal series of $SU(1,1)$, or the values $S < -1/2$, that correspond to the discrete series of the group. At S integer or half-integer the representations are single-valued. We demand $\psi_u^+(x,z)$ to be an eigenfunction for the Lubanski–Pauli operator $\hat{W} = \hat{p}\hat{S}$,

$$\hat{W}\psi_u^+(x,z) = ms\psi_u^+(x,z). \tag{5.27}$$

The left-hand side of (5.27) takes the form, after the action of the operator $\hat{W}$,

$$\begin{aligned} &S(\hat{p}_0(\bar{z}_2u^2 - z_1u^1) - \hat{p}_1(z_1u^2 - \bar{z}_2u^1) - \mathrm{i}\hat{p}_2(z_1u^2 + \bar{z}_2u^1))(z_1u^1 + \bar{z}_2u^2)^{2S-1} \\ &\qquad = S(\bar{z}_2z_1)p_\mu\gamma^\mu \begin{pmatrix} u^2(x) \\ u^1(x) \end{pmatrix} (z_1u^1 + \bar{z}_2u^2)^{2S-1}. \end{aligned}$$

Thus we obtain an equation for the parameters of CS (5.26),

$$\left(\hat{p}_\mu\gamma^\mu - \frac{s}{S}m\right)\begin{pmatrix} u^2(x) \\ u^1(x) \end{pmatrix} = 0 \tag{5.28}$$

which is a 2 + 1 Dirac equation with mass $m' = (s/S)m$. The same equation controls the evolution of the parameters of CS (5.25), and also appears both in the case $S = -1/2$, and for arbitrary $S < -1/2$.

5.3. IR of $\tilde{M}(2,1)$: classification and bases

Here we are going to derive explicit forms of eigenfunctions for sets of commuting operators of $\tilde{M}(2,1)$, decomposing GRR in IRs. A classification and a description of the unitary IR of the group will also be given.

It is possible to construct bases for particles with spin, which consist of eigenvectors for different sets of commuting operators. For example, for sets of operators: ($\hat{p}_\mu$, $\hat{W}$, $\hat{\boldsymbol{S}}^2$), ($\hat{\boldsymbol{p}}^2$, $\hat{W}$, $\hat{\boldsymbol{S}}^2$, $\hat{\boldsymbol{J}}^2$, J^0), ($\underline{\hat{p}}_\mu$, $\hat{W}$, $\hat{\boldsymbol{J}}^2$), ($\hat{\boldsymbol{p}}^2$, $\hat{\boldsymbol{S}}^2$, $\hat{p}_0$, $\hat{L}^0$, $\hat{S}^0$ (we did not include the Casimir operator $\hat{W}$ in this set since it does not commute with the operators $\hat{L}^\mu$ and $\hat{S}^\mu$ separately)), ($\hat{p}_\mu$, $\underline{\hat{p}}_\mu$, $\hat{W}$), and so on.

Let us consider states, which are eigenvectors for the operators $\hat{p}_\mu$, $\hat{W}$, $\hat{\boldsymbol{S}}^2$ (plane waves). They can be written in the following form

$$f_{p,S}(x,z) = \mathrm{e}^{-\mathrm{i}px} f_S(p,z) \tag{5.29}$$

where $f_S(z)$ is a homogeneous function on the variables z_1, $\bar{z}_2$ of the power $2S$. These states are important to classify IRs of $\tilde{M}(2,1)$ by means of the little group method.

It is known that IRs of the motion groups of the pseudo-Euclidean spaces (Poincare groups) are marked completely by means of parameters of orbits in the space of momenta and by numbers, which characterize the IRs of a stationary subgroup of a state, belonging to the orbit (little group) [11]. Thus let us consider three cases: $m > 0$ (orbits O_m^+, O_m^-), $m = 0$ (orbits O_0^+, O_0^-, O_0^0), and $m^2 < 0$ (orbits O_m).

(1) At $m > 0$, in the rest frame, $\hat{p}\hat{S} = \pm m\hat{S}^0$, so that the eigenvectors of this operator with the eigenvalues $\pm ms$ are

$$f_{p,S}(x,z) = \mathrm{e}^{-\mathrm{i}p_0x^0}\bar{z}_2^{S+s}(-z_1)^{S-s}. \tag{5.30}$$

One can find the stationary subgroup of the state (5.30) from the condition $U^{-1}P_0(U^{-1})^\dagger = P_0$, where $P_0 = \mathrm{diag}(m, m)$. The matrices $U = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\varphi/2}, \mathrm{e}^{\mathrm{i}\varphi/2})$ obey the condition and form a one-parametric subgroup, which is isomorphic to the group $U(1)$ with the generator $\hat{J}^0 = \hat{L}^0 + \hat{S}^0$. The eigenvalues s of this operator together with the characteristic of the orbit mark IR of $\tilde{M}(2,1)$. Let us denote such representations as $T^+_{m,s}$ and $T^-_{m,s}$. They are single-valued at s integer and half-integer, whereas ms and $-ms$ are the eigenvalues of the operator $\hat{p}\hat{S}$ in these representations, respectively. Subjecting the state (5.30) to a finite transformation of $\tilde{M}(2,1)$, we get the function

$$f_{p',S,s}(x,z) = \mathrm{e}^{-\mathrm{i}p'x} N_{S,s}(\bar{z}_2u_1 - z_1\bar{u}_2)^{S+s}(\bar{z}_2u_2 - z_1\bar{u}_1)^{S-s} \qquad P' = U^{-1}P_0(U^{-1})^\dagger. \tag{5.31}$$

The spinning part of the function is the CS of $SU(1,1)$. The parameters $u_1, \bar{u}_2$ are expressed via the momentum p' (see (5.21)). This function describes a particle with real mass $m \neq 0$, momentum p', Lorentz spin S, and the spin projection s. The normalization coefficient $N_{S,s}$ depends on IR series, see the appendix.

The wavefunction of a massive particle with Lorentz spin S, energy p_0, angular momentum projection l, and spin projection ζ on the axis x^0, has the form, according (3.30) of

$$f_{p_0,S,\zeta,l}(x,z) = \mathrm{e}^{-\mathrm{i}p_0x^0+\mathrm{i}l\phi} J_l\left(\rho\sqrt{p_0^2 - m^2}\right) N_{S,\zeta}\bar{z}_2^{S+\zeta}(-z_1)^{S-\zeta}. \tag{5.32}$$

(2) The wavefunction of a massless particle with $p^\mu = p(1,1,0)$ is

$$f_{p,S}(x,z) = \mathrm{e}^{-\mathrm{i}p(x^0-x^1)} f_S(z) \qquad \hat{W} f_{p,S}(x,z) = p\,\mathrm{e}^{-\mathrm{i}p(x^0-x^1)}(\hat{S}^0 - \hat{S}^1) f_S(z).$$

The operator $\hat{S}^0 - \hat{S}^1$ is the generator of the stationary subgroup of the state. The U matrices, which correspond to the subgroup, obey the condition

$$U^{-1}P_{01}(U^{-1})^\dagger = P_{01} \qquad P_{01} = \begin{pmatrix} p & p \\ p & p \end{pmatrix}$$

and have the form

$$U = \pm\begin{pmatrix} 1+\mathrm{i}a & \mathrm{i}a \\ -\mathrm{i}a & 1-\mathrm{i}a \end{pmatrix}.$$

They form an $R \otimes Z$ group, where R is the additive group of the real numbers, and Z is the multiplicative group, which consist of two elements $\{1, -1\}$. These two elements correspond to the identical transformation and to $\varphi = 2\pi$ rotation around the axis x^0, respectively $U = I$ and $U = -I$, where I is the unit matrix. One can see from (4.4) that the latter rotation does not change x but changes the sign of z, $T(2\pi)f(x,z) = f(x,-z)$.

The eigenvectors of the operator $\hat{S}^0 - \hat{S}^1$, which correspond to the eigenvalues λ, have the form

$$f_\lambda(z) = F(z_1 - \bar{z}_2)\exp\left(\lambda\frac{z_1 + \bar{z}_2}{\bar{z}_2 - z_1}\right). \tag{5.33}$$

The wavefunctions of a massless particle with the momentum $(p, p, 0)$, Lorentz spin S, and the spin projection λ on the direction of the momentum can be written as

$$f_{p,S,\lambda}(x,z) = \mathrm{e}^{-\mathrm{i}p(x^0-x^1)}(z_1 - \bar{z}_2)^{2S}\exp\left(\lambda\frac{z_1 + \bar{z}_2}{\bar{z}_2 - z_1}\right). \tag{5.34}$$

They are eigenvectors of the operators $\hat{W}$ and $\hat{\boldsymbol{S}}^2$ with the eigenvalues $K = p\lambda$ and $S(S+1)$. These functions change sign under the Z-transformations (rotations on 2π) at half-integer S and remain unchanged at S integer. We denote IRs, which correspond to $m = 0$, by $T^+_{0,\varepsilon,K}$ and $T^-_{0,\varepsilon,K}$. Here $\varepsilon = 0$ (S integer) or $\varepsilon = 1$ (S half-integer) mark IRs of the Z group. One can see that $(\hat{S}^0-\hat{S}^1)^n = [(\bar{z}_2-z_1)(\partial/\partial z_1+\partial/\partial\bar{z}_2)/2]^n = [(\bar{z}_2-z_1)/2]^n(\partial/\partial z_1+\partial/\partial\bar{z}_2)^n$, and, therefore, the operator $\hat{S}^0 - \hat{S}^1$ can only have zero eigenvalues in the space of polynomials. Thus as was mentioned in [3], eigenvalues of the Casimir operator $\hat{W}$ are zero for the finite-dimensional in spin wavefunctions of the massless particles. That can be seen directly using the explicit form of the states (5.33) and (5.34). At $\lambda \neq 0$ there is an exponential factor dependent on z, its z-decomposition leads to an infinite number of wavefunction components, similar states appear in the tachyon case.

Table 1. Unitary single-valued IRs of $\tilde{M}(2, 1)$.

Mass orbits	IR	Eigenvalue $\hat{W} = \hat{p}\hat{S}$	States	Remarks
$m > 0$,	$T^+_{m,s}$	ms	(5.31)	$s \geqslant 0$, integer or half-integer
O^+_m, O^-_m	$T^-_{m,s}$	$-ms$		
$m = 0$,	$T^+_{0,\varepsilon}$	0	(5.35)	$\varepsilon = 0, 1$
O^+_0, O^-_0	$T^-_{0,\varepsilon}$	0		
	$T^+_{0,K,\varepsilon}$	$K = p\lambda$	(5.34)	$K \neq 0$, real, infinite-dimensional IR
	$T^-_{0,K,\varepsilon}$	$K = p\lambda$		
$m^2 < 0$,	$T_{m,0,\varepsilon}$	0	(5.36)	
O_m	$T_{m,\sigma,\varepsilon}$	$\mathrm{i}m\sigma$	(5.36)	$\sigma \neq 0$, real, infinite-dimensional IR
$m = 0$,	T^+_S, T^-_S	0	See the	Discrete series of $SU(1, 1)$
O^0_0	$T_{S,\varepsilon}$	0	appendix	Principal series of $SU(1, 1)$
	T_S	0		Supplementary series of $SU(1, 1)$
	T^0_0	0		Invariant

At $\lambda = 0$, $f_S(z) = (z_1 - \bar{z}_2)^{2S}$ and if $S \geqslant 0$ integer or half-integer, then the number of components is finite (is equal to $2S + 1$). We denote IRs at $\lambda = 0$ by $T^+_{0,\varepsilon}$ and $T^-_{0,\varepsilon}$, where $\varepsilon = 0$ corresponds to the integer and $\varepsilon = 1$ to half-integer S. The case of an arbitrary direction of movement, $p'^\mu = p(1, \cos\varphi, \sin\varphi)$, can be derived by a rotation around the axis x^0, $U = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi/2}, \mathrm{e}^{-\mathrm{i}\varphi/2})$, then $z'_1 = z_1\,\mathrm{e}^{-\mathrm{i}\varphi/2}$, $\bar{z}'_2 = \bar{z}_2^{\mathrm{i}\varphi/2}$. In particular, at $\lambda = 0$,

$$f_{p',S}(x, z) = \mathrm{e}^{-\mathrm{i}p'x}(z_1\,\mathrm{e}^{-\mathrm{i}\varphi/2} - \bar{z}_2\,\mathrm{e}^{\mathrm{i}\varphi/2})^{2S}. \tag{5.35}$$

This function describes a massless particle with the momentum p' and Lorentz spin S.

(3) In the case of tachyons, the state with $p_0 = p_2 = 0$, $p_1 = \mathrm{i}m$,

$$f_{p,S}(x, z) = \mathrm{e}^{-\mathrm{i}p_1x^1} f_S(z)$$

has the stationary subgroup, which can be found from the condition $U^{-1}P_1(U^{-1})^\dagger = P_1$, where

$$U = \pm\begin{pmatrix} \cosh\theta/2 & \mathrm{i}\sinh\theta/2 \\ -\mathrm{i}\sinh\theta/2 & \cosh\theta/2 \end{pmatrix} \qquad P_1 = \begin{pmatrix} 0 & -\mathrm{i}m \\ -\mathrm{i}m & 0 \end{pmatrix}.$$

This subgroup is isomorphic to $R \otimes Z$ and has the generator $\hat{J}^1$. The eigenvectors $f_{p,S}(x, z)$ for the operators $\hat{S}^1$ and $\hat{\boldsymbol{S}}^2$, with the eigenvalues σ and $S(S + 1)$ respectively, have the form

$$f_{p,S,\sigma}(x, z) = \mathrm{e}^{-\mathrm{i}p_1x^1}(\bar{z}_2 + \mathrm{i}z_1)^{S+\mathrm{i}\sigma}(\bar{z}_2 - \mathrm{i}z_1)^{S-\mathrm{i}\sigma} = \mathrm{e}^{-\mathrm{i}p_1x^1}(\bar{z}_2^2 + z_1^2)^S\left(\frac{\bar{z}_2 - \mathrm{i}z_1}{\bar{z}_2 + \mathrm{i}z_1}\right)^{-\mathrm{i}\sigma}. \tag{5.36}$$

Functions $f_{p,S,\sigma}(x,z)$ are the eigenvectors for the Casimir operators $\hat{W}$ and $\hat{\boldsymbol{p}}^2$ with the eigenvalues $p_1\sigma$ and $-p_1^2$, respectively. σ has to be real for unitary IRs, therefore, for $\sigma \neq 0$, representations, which correspond to the imaginary mass case, are infinite-dimensional in the spin. The case of arbitrary direction of the momentum can be derived by means of a rotation, as was done above for the real and zero mass.

(4) Unitary IRs of $\tilde{M}(2,1)$, which are connected with the orbit O_0^0, are IRs of $SU(1,1)$.

The classification of the single-valued unitary IR of the $\tilde{M}(2,1) = T(3)\times)SU(1,1)$ group are summarized in table 1.

The IR states of $SU(1,1)$, corresponding to the orbit O_0^0, do not depend on x and are invariant under translations. The sign ($+$ or $-$) at T is related to the sign of p_0. The characteristic 'infinite-dimensional' means infinite-dimensionality in the spin space.

The finite-dimensional spin wavefunctions of massless particles and tachyons are zero modes of the operator $\hat{W}$.

To complete the picture one has to add to this table multivalued representations $T_{m,s}^+$ and $T_{m,s}^-$ at non-integer $2s$, and multivalued IRs of $SU(1,1)$, described in the appendix. The explicit form of states, which are transformed under the representations $T_{m,s}^+$ and $T_{m,s}^-$ at non-integer $2s$, can also be given by the formula (5.31), however, in this case, z-decomposition generates an infinite number of components. Just those IRs are used to describe anyons.

Appendix. Unitary IR and coherent states of the $SU(1,1)$ group

The $2+1$ Lorentz group $SO(2,1)$, and closely related groups $SU(1,1)$ and $SL(2,R)$ with the same algebra, have been studied in numerous papers [13–17, 22, 26–40]. Their finite-dimensional IRs and unitary IRs (discrete series) are used to describe spin in $(2+1)$-dimensions [4]. As is known, $SO(3,1)$ has only principal and supplementary series of unitary representations, and the principal series is used to describe spin in $3+1$ dimensions [41, 42]. In this connection, in spite of everything, it is important to consider the same series of $SO(2,1)$ or $SU(1,1)$.

We are going to describe unitary IRs of $SU(1,1)$, their discrete bases and corresponding CS. The consideration, to be complete, is going to repeat some known results, but also to present some new ones. For example, we are constructing CS in unitary IRs of the principal series at arbitrary fractional projections of the angular momentum in addition to [39], where only integer ones were considered. We construct unitary IRs, including multivalued, in spaces of functions on various manifolds connected with $SO(2,1)$ or $SU(1,1)$, whereas usually they are restricted to the unit disk or to a circle. In particular, we consider decompositions of functions on a cone and one-sheeted hyperboloids with respect to unitary IRs of $SO(2,1)$.

Consider the left representation $T(U)$, $U \in SU(1,1)$, acting in the space of functions $f(v)$,

$$T(U)f(v) = f(U^{-1}v) \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{A1}$$

The matrices U^{-1} can be parametrized by two complex numbers u^1, u^2,

$$U^{-1} = \begin{pmatrix} \bar{u}_1 & -u_2 \\ -\bar{u}_2 & u_1 \end{pmatrix} \qquad |u^1|^2 - |u^2|^2 = 1. \tag{A2}$$

The combination

$$|v_1|^2 - |v_2|^2 = C \tag{A3}$$

remains invariant under the $SU(1,1)$ transformations. Generators $\hat{J}^\mu$, which correspond to one-parametrical subgroups with parameters $-\alpha^\mu$ (see (2.3)), and arising $\hat{J}_+$ and lowering $\hat{J}_-$ operators have the following form in this representation

$$\begin{aligned}
&\hat{J}^0 = -(1/2)(v_1\partial/v_1 - v_2\partial/v_2) \qquad \hat{J}_- = v_1\partial/v_2 \qquad \hat{J}_+ = v_2\partial/v_1\\
&\hat{J}^1 = (1/2)(\hat{J}_+ - \hat{J}_-) = (1/2)(v_2\partial/v_1 - v_1\partial/v_2)\\
&\hat{J}^2 = (\mathrm{i}/2)(\hat{J}_+ + \hat{J}_-) = (\mathrm{i}/2)(v_1\partial/v_2 + v_2\partial/v_1).
\end{aligned} \tag{A4}$$

They obey the commutation relations

$$[\hat{J}^\mu, \hat{J}^\nu] = -\mathrm{i}\epsilon^{\mu\nu\eta}\hat{J}_\eta \qquad [\hat{J}_+, \hat{J}_-] = 2\hat{J}^0 \qquad [\hat{J}^0, \hat{J}_\pm] = \pm\hat{J}_\pm$$

so that $\hat{\boldsymbol{J}}^2$ is a Casimir operator,

$$\begin{aligned}
\hat{\boldsymbol{J}}^2 &= \hat{J}_\mu\hat{J}^\mu = (\hat{J}_0)^2 + \tfrac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)\\
&= \tfrac{1}{4}(v_1\partial/\partial v_1 + v_2\partial/\partial v_2)(v_1\partial/\partial v_1 + v_2\partial/\partial v_2 + 2).
\end{aligned}$$

Let us take functions of the form $f_{n_1n_2}(v) = v_1^{n_1}v_2^{n_2}$. The action of the generators on these functions can be found†,

$$\begin{aligned}
&\hat{J}^0 f_{n_1n_2} = m f_{n_1n_2} \qquad \hat{\boldsymbol{J}}^2 f_{n_1n_2} = j(j+1) f_{n_1n_2} \qquad m = \frac{n_2 - n_1}{2} \qquad j = \frac{n_1 + n_2}{2}\\
&\hat{J}_- f_{n_1n_2} = n_2 f_{n_1+1,n_2-1} \qquad \hat{J}_+ f_{n_1n_2} = n_1 f_{n_1-1,n_2+1}.
\end{aligned} \tag{A5}$$

Thus, quasi-polynomials of the power $2j$ form an IR space (j characterizes the IR). $\hat{J}_+$ and $\hat{J}_-$ are arising and lowering operators for the projection of the angular momentum $m = (n_2 - n_1)/2$. If $n_2 \geqslant 0$ and is integer then $f_{n_1n_2}$ belongs to an IR which has the lowest weight v_1^{2j}; if $n_1 \geqslant 0$, and is integer then the IR has the highest weight v_2^{2j}; if both $n_i \geqslant 0, i = 1, 2$, and are integer then the IR is finite-dimensional (has both the highest and lowest weights). For unitary IRs of $SU(1,1)$: $(\hat{J}^0)^+ = \hat{J}^0$, $\hat{J}^+_\pm = -\hat{J}_\mp$, that means $n_2 - n_1$ is real, and $n_1(n_2+1) \leqslant 0$, $n_2(n_1+1) \leqslant 0$, whereas for the IR of $SU(2)$: $\hat{J}^+_\pm = \hat{J}_\mp$ and $n_1(n_2+1) \geqslant 0$, $n_2(n_1+1) \geqslant 0$ [22]. At a given j one can select

$$N_{n_1n_2} v_1^{n_1} v_2^{n_2} \tag{A6}$$

as elements of a discrete basis in the space of functions $f_j(v)$, where $N_{n_1n_2}$ is the normalization constant, and $n_1 = j - m$, $n_2 = j + m$.

A classification and weight structure of unitary infinite-dimensional and non-unitary finite-dimensional IRs of $SU(1,1)$ is presented in figure A1.

To describe the IRs of different series one has to define in more detail the space of functions $f(v)$. At different C in (A3) one can use the following parametrization of v_1 and v_2:

$$\begin{aligned}
&C = 0\colon v_1 = \rho\,\mathrm{e}^{\mathrm{i}(\varphi+\omega)/2} \qquad v_2 = \rho\,\mathrm{e}^{\mathrm{i}(\omega-\varphi)/2}\\
&0 < \rho < +\infty \qquad 0 \leqslant \varphi < 4\pi \qquad 0 \leqslant \omega < 2\pi
\end{aligned} \tag{A7}$$

$$\begin{aligned}
&C = 1\colon v_1 = \cosh(\theta/2)\,\mathrm{e}^{\mathrm{i}(\varphi+\omega)/2} \qquad v_2 = \sinh(\theta/2)\,\mathrm{e}^{\mathrm{i}(\omega-\varphi)/2}\\
&0 \leqslant \theta < +\infty \qquad 0 \leqslant \varphi < 4\pi \qquad 0 \leqslant \omega < 2\pi.
\end{aligned} \tag{A8}$$

The case of negative C ($C = -1$) is reduced to (A8) by the replacement $v_1 \leftrightarrow v_2$. The parameter ω is not changed under the group transformations in the case (A7), thus, there are

† We are going to use here the notation m for the anglular momentum projection (the same was used for the mass), hoping that this will not lead to a misunderstanding.

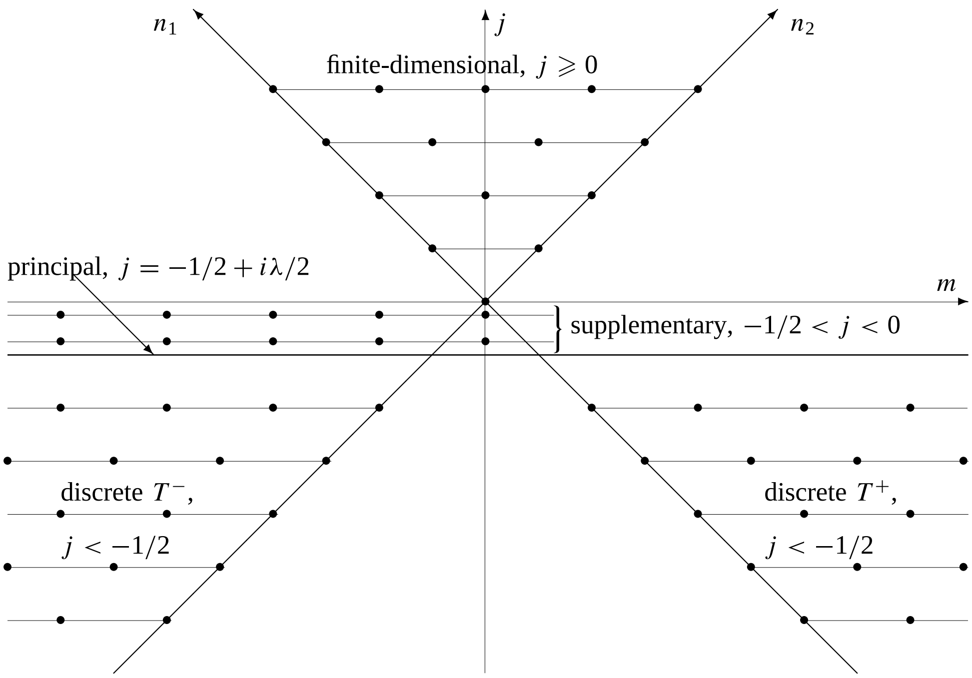

Figure A1. Weight diagrams for unitary and finite-dimensional IRs of $SU(1, 1)$.

two complex manifolds, on which the group is acting transitive: the complex hyperboloid (A8) and the cone,

$$C = 0\colon v_1 = \rho\, \mathrm{e}^{\mathrm{i}\varphi/2} \qquad v_2 = \rho\, \mathrm{e}^{-\mathrm{i}\varphi/2} \qquad 0 < \rho < +\infty \qquad 0 \leqslant \varphi < 4\pi. \tag{A9}$$

Using the components (v_1, v_2) of the spinor and the complex conjugate components $(\bar{v}_1, \bar{v}_2)$, one can construct objects (x^0, x^1, x^2), which are transformed under the three-dimensional vector IRs with $j = 1$,

$$x^0 = (|v_1|^2 + |v_2|^2)/2 \qquad x^1 = (\bar{v}_1 v_2 + v_1 \bar{v}_2)/2 \qquad x^2 = (v_1 \bar{v}_2 - \bar{v}_1 v_2)/2\mathrm{i} \tag{A10}$$

$$x^0 = v_1 v_2 \qquad x^1 = (v_1^2 + v_2^2)/2 \qquad x^2 = (v_1^2 - v_2^2)/2\mathrm{i}. \tag{A11}$$

The vectors (A10) and (A11) have the same transformation properties, since the spinors (v_1, v_2) and $(\bar{v}_2, \bar{v}_1)$ are transformed equally. The latter can be easily checked, using the explicit form of the matrix (A2). Substituting (A9) into (A10) or (A11), we get the cone

$$x^0 = \rho^2 \qquad x^1 = -\rho^2 \cos\varphi \qquad x^2 = -\rho^2 \sin\varphi \qquad x_0^2 - x_1^2 - x_2^2 = 0. \tag{A12}$$

Substituting (A8) into (A10), we get the two-sheeted hyperboloid

$$x^0 = \cosh\theta \qquad x^1 = -\sinh\theta \cos\varphi \qquad x^2 = -\sinh\theta \sin\varphi \qquad x_0^2 - x_1^2 - x_2^2 = 1. \tag{A13}$$

If v_k are periodic in φ with the period 4π, then x_μ are also periodic with the period 2π.

Let us turn first to IRs of the discrete series $T_j^+(m = -j, -j+1, -j+2, \ldots)$ and $T_j^-(m = j, j-1, j-2, \ldots)$, $j < -1/2$, the theory of which is quite similar to that of the finite-dimensional IR. The IR T_j^+ and T_j^- can be realized in the space of functions $f(v)$, where v_1 and v_2 belong to the case (A8). The scalar product of functions on the complex hyperboloid,

$$\langle f_1 \mid f_2\rangle = \frac{1}{8\pi^2}\int \overline{f}_1 f_2 \delta(|v_1|^2 - |v_2|^2 - 1)\,\mathrm{d}^2v_1\,\mathrm{d}^2v_2$$
$$= \frac{1}{8\pi^2}\int_0^{2\pi}\mathrm{d}\omega\int_{-2\pi}^{2\pi}\mathrm{d}\varphi\int_0^{\infty}\overline{f}_1 f_2 \sinh\theta\,\mathrm{d}\theta\,\mathrm{d}^2v = \mathrm{d}\Re v\,\mathrm{d}\Im v \quad \text{(A14)}$$

allows one to normalize the elements of the discrete basis T_j^+ at $j < -1/2$,

$$\psi_{j,m}(v) = \langle v \mid jm\rangle = \left(\frac{(-1)^{n_2}\Gamma(-n_1)}{n_2!\Gamma(-2j)}\right)^{1/2} v_1^{n_1}v_2^{n_2}$$
$$= \left(\frac{(-1)^{n_2}\Gamma(-n_1)}{n_2!\Gamma(-2j)}\right)^{1/2}(\cosh(\theta/2))^{n_1}(\sinh(\theta/2))^{n_2}\,\mathrm{e}^{\mathrm{i}m(\varphi+4\pi k)}\,\mathrm{e}^{\mathrm{i}j(\omega+2\pi k)}. \quad \text{(A15)}$$

The projection m, and therefore $j(j = m_{\max}$ in T_j^-, $j = -m_{\min}$ in $T_j^+)$, have to run over the integer and half integer, $j = -1, -3/2, -2, \ldots,$ for representations in spaces of single-valued functions.

The lowest weight $\langle v \mid j\ -j\rangle = v_2^{2j}$ has a stationary subgroup $U(1)$ and CS are parametrized by dots of the upper sheet of the two-sheeted hyperboloid $SU(1,1)/U(1)$. An explicit form of CS can be obtained by the action of finite transformations on the lowest weight,

$$\psi_{j,u}(v) = \langle v \mid ju\rangle = (\bar{u}_1 v_1 + u_2 v_2)^{2j} \quad \text{(A16)}$$

where $u = (\bar{u}_1, -u_2)$, $\bar{u}_1 = \cosh(\theta_1/2)\,\mathrm{e}^{\mathrm{i}m\varphi_1/2}$, $-u_2 = \sinh(\theta_1/2)\,\mathrm{e}^{-\mathrm{i}m\varphi_1/2}$ are elements of the matrix (A2). The CS overlapping has the form

$$\langle j'u' \mid ju\rangle = \delta_{jj'}(u_1'\bar{u}_1 - \bar{u}_2'u_2)^{2j}. \quad \text{(A17)}$$

A detailed description of CS of the discrete series of $SU(n,1)$ can be found in [40], and of $SU(1,1)$ in [22, 39, 40]. The representations T_j^+ and T_j^- are conjugate; the discrete basis T_j^- can be derived by means of the complex conjugation from (A15) or by the replacement $v_1 \leftrightarrow v_2$.

For the functions, which are transformed with respect to one and the same representation T_j^+, the integral over ω in (A14) gives 2π. The completeness relation at a given j can be written both in terms of the discrete basis and in terms of CS,

$$\hat{1}_j = \sum_{m=-\infty}^{j}|jm\rangle\langle jm| = \frac{-2j-1}{4\pi}\int_{-2\pi}^{2\pi}\mathrm{d}\varphi_1\int_0^{\infty}|j\theta_1\varphi_1\rangle\langle j\theta_1\varphi_1|\sinh\theta_1\,\mathrm{d}\theta_1. \quad \text{(A18)}$$

The parameter j takes discrete values and the basis functions are orthonormalized on the Kronecker symbol $\delta_{jj'}$ for the single-valued IR of the discrete series, whereas for the principal series the condition of the orthonormality contains the δ-function $\delta(j - j')$. Principal series can be constructed both in the space of functions on the complex hyperboloid (A8), and on the cone (A9).

One can construct the principal series on the cone (A9) with the scalar product

$$\langle f_1 \mid f_2\rangle = (1/8\pi^2)\int_{-2\pi}^{2\pi}\mathrm{d}\varphi\int_0^{\infty}\overline{f_1(\rho,\varphi)}\,f_2(\rho,\varphi)\rho\,\mathrm{d}\rho. \quad \text{(A19)}$$

We get $C_{n_1n_2} = 1$, $n_1 + n_2 = 2j = -1 + \mathrm{i}\lambda$, $2m = n_2 - n_1$, for the elements of the discrete basis (A6) in the case of the principal series,

$$\hat{J}_\pm = \mathrm{e}^{\pm\mathrm{i}\varphi}((1/2)\rho\partial/\partial\rho \pm \mathrm{i}\partial/\partial\varphi) \qquad \hat{J}_0 = -\mathrm{i}\partial/\partial\varphi \quad \text{(A20)}$$
$$\langle\rho\varphi \mid \lambda m\rangle = v_1^{n_1}v_2^{n_2} = \rho^{-1+\mathrm{i}\lambda}\,\mathrm{e}^{\mathrm{i}m(\varphi+4\pi k)} \qquad \langle\lambda m \mid \lambda'm'\rangle = \delta(\lambda-\lambda')\delta_{mm'}$$
$$\langle\rho\varphi \mid \rho'\varphi'\rangle = (1/\rho\rho')\delta(\ln\rho - \ln\rho')\delta(\varphi-\varphi') = (1/\rho)\delta(\rho-\rho')\delta(\varphi-\varphi'). \quad \text{(A21)}$$

Two IRs in the space of single-valued functions (with integer and half-integer m, the first and the second principal series accordingly to the terminology of the work [15]) correspond to each given λ,

$$\hat{1} = \frac{1}{8\pi^2}\int_{-2\pi}^{2\pi} \mathrm{d}\varphi \int_0^\infty |\rho\varphi\rangle\langle\rho\varphi|\rho\,\mathrm{d}\rho = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} \mathrm{d}\lambda \sum_m |\lambda m\rangle\langle\lambda m|.$$

The summation in the last equation runs over all integer and half-integer m. Multivalued IRs are characterized not only by λ but also by a number ε, $|\varepsilon| \leqslant 1/2$, which gives the nearest-to-zero value of m (for single-valued IR, $\varepsilon = 0$ or $\varepsilon = \pm 1/2$). Elements of the infinite-valued IR space are not periodic in φ. Thus, an arbitrary representation of the principal series is defined by two numbers (λ, ε), where $j = (-1 + \mathrm{i}\lambda)/2$ characterizes the angular momentum square, $\boldsymbol{J}^2 = j(j+1) = (-1-\lambda^2)/4$, and ε characterizes possible values of the momentum projection $m = \varepsilon + [m]$. There is a certain analogy with IRs of the principal series of $SO(3,1)$, which are defined by two numbers (λ, S), where S corresponds to the spin [41, 42], and λ defines the square of the four-dimensional angular momentum.

The representation of the principal series $T_{-1/2}$ is reducible at $\lambda = 0$ and $|\varepsilon| = 1/2$, and is split into two IRs: $T^+_{-1/2}(\varepsilon = -1/2)$ and $T^-_{-1/2}(\varepsilon = 1/2)$; $\varepsilon = \pm 1/2$ corresponds to one and the same IR at $\lambda \neq 0$.

One can remark that according to (A21), the ρ-dependence of functions on the cone is the same at a fixed j, and it is possible to consider the space of functions $f(\varphi)$ on the circle, what they are usually doing, by considering the principal series of IRs. However, such a reduction of the representation space is not always reasonable because the space of functions on the cone sometimes appears naturally in different physical problems.

To construct CS one has to consider orbits in the representation space, factorized with respect to stationary subgroups [39]. The stationary subgroup of the state $|\lambda m = 0\rangle = \rho^{-1+\mathrm{i}\lambda}$ is $U(1)$, and CS, which correspond to integer $m(\varepsilon = 0)$, are parametrized by the dots (θ, ψ) on the upper sheet of the hyperboloid $SU(1,1)/U(1)$. (Such CS were constructed in [39, 43] in the space of functions on a circle.) Substituting $\bar{u}_1 = \cosh(\theta/2)\,\mathrm{e}^{\mathrm{i}\psi/2}$, $-u_2 = \sinh(\theta/2)\,\mathrm{e}^{-\mathrm{i}\psi/2}$, $\rho' = \rho(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{1/2}$ in (A1) and (A2), we get CS in the form

$$\langle\rho\varphi \mid \lambda\theta\psi\rangle = (\rho')^{-1+\mathrm{i}\lambda} = \rho^{-1+\mathrm{i}\lambda}(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{-1/2+\mathrm{i}\lambda/2}$$

$$\begin{aligned}\langle\lambda m \mid \lambda_1\theta\psi\rangle &= \frac{1}{8\pi^2}\int\int \langle\lambda m \mid \rho\varphi\rangle\langle\rho\varphi \mid \lambda_1\theta\psi\rangle\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= (1/2\pi)\delta(\lambda-\lambda_1)\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\varphi}(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{-1/2+\mathrm{i}\lambda/2}\,\mathrm{d}\varphi \\ &= \delta(\lambda-\lambda_1)\frac{\Gamma(m+1)}{\Gamma(m+1/2+\mathrm{i}\lambda/2)} P^m_{-1/2+\mathrm{i}\lambda/2}(\cosh\theta)\,\mathrm{e}^{-\mathrm{i}m\psi}\end{aligned} \tag{A22}$$

where $P^m_{-1/2+\mathrm{i}\lambda/2}(\cosh\theta)$ is adjoint Legendre function. At $m = 0$ the latter goes over to zonal harmonic $P_{-1/2+\mathrm{i}\lambda/2}(\cosh\theta)$ (it is also called the cone function [15, 44]). To get CS at arbitrary ε one has to act by means of finite transformations on the state $|\lambda m = \varepsilon\rangle = \rho^{-1+\mathrm{i}\lambda}\,\mathrm{e}^{\mathrm{i}\varepsilon\varphi}$,

$$\begin{aligned}\langle\rho\varphi \mid \lambda\varepsilon\theta\psi\rangle &= ((v_1\bar{u}_1 - v_2u_2)(-v_1\bar{u}_2 + v_2u^1))^{-1/2+\mathrm{i}\lambda/2}\left(\frac{-v_1\bar{u}_2 + v_2u_1}{v_1\bar{u}_1 - v_2u_2}\right)^{\varepsilon} \\ &= \rho^{-1+\mathrm{i}\lambda}(\cosh\theta + \sinh\theta\cos(\varphi+\psi))^{-1/2+\mathrm{i}\lambda/2} \\ &\quad\times\left(\frac{\cosh(\theta/2)\exp[-\mathrm{i}(\varphi-\psi)/2] + \sinh(\theta/2)\exp[\mathrm{i}(\varphi-\psi)/2]}{\cosh(\theta/2)\exp[\mathrm{i}(\varphi-\psi)/2] + \sinh(\theta/2)\exp[-\mathrm{i}(\varphi-\psi)/2]}\right)^{\varepsilon}.\end{aligned} \tag{A23}$$

The case $\varepsilon = 0$, which we have already considered, and $\varepsilon = \pm 1/2$, correspond to representations in spaces of single-valued functions. In the latter case at $m = \pm 1/2$ we get

$$\langle \rho\varphi \mid \lambda 1/2, \theta\psi\rangle = (v_1\bar{u}_1 - v_2 u_2)^{-1} |v_1\bar{u}_1 - v_2 u_2|^{\mathrm{i}\lambda}$$
$$\langle \rho\varphi \mid \lambda\, -1/2, \theta\psi\rangle = (-v_1\bar{u}_2 + v_2 u_1)^{-1} |v_1\bar{u}_1 - v_2 u_2|^{\mathrm{i}\lambda}. \qquad \text{(A24)}$$

At $\lambda = 0$ the CS take a simple form

$$\langle \rho\varphi \mid 0\, 1/2, \theta\psi\rangle = (v_1\bar{u}_1 - v_2 u_2)^{-1} \qquad \langle \rho\varphi \mid 0\, -1/2, \theta\psi\rangle = (-v_1\bar{u}_2 + v_2 u_1)^{-1} \qquad \text{(A25)}$$

which coincides with the explicit form of CS of the discrete series (A16) (in this case, the only difference between the CS of different series consists of different domains of v_1 and v_2, see (A8) and (A9)).

Let us turn to IRs of supplementary series. The integral in (A19) is divergent at real j. However, one can use a convergent 'non-local' scalar product

$$\langle f_1 \mid f_2\rangle = \int\int \overline{f_1(x_1)}\, f_2(x_2) I(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \qquad \text{(A26)}$$

where the kernel function $I(x_1, x_2)$ has to be invariant with respect to the group transformations. For the cone one can select an invariant expression $(v_1\bar{v}_1' - \bar{v}_2 v_2') = 2\mathrm{i}\sin(\varphi/2 - \varphi'/2)\rho\rho'$. At a fixed j representation functions have the form $\rho^{2j} f(\varphi)$. Let us select $I(x_1, x_2) = |(v_1 v_1' - v_2 v_2')/2|^{-2j}$, then the integrand in (A26) is $\overline{f_1(\varphi)}\, f_2(\varphi')|\sin(\varphi/2 - \varphi'/2)|^{-2j}$. It does not depend on ρ, so that at a fixed j (A26) takes the form

$$\langle f_1 \mid f_2\rangle = \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi} \overline{f_1(\varphi)}\, f_2(\varphi')|\sin(\varphi/2 - \varphi'/2)|^{-2j}\, \mathrm{d}\varphi\, \mathrm{d}\varphi' \qquad \text{(A27)}$$

where $-1/2 < j < 0$, the latter is necessary for the scalar product to be convergent and positive defined.

For the single-valued representations of the supplementary series m is integer, for the multi-valued representations one has to introduce ε, $|\varepsilon| \leqslant |j|$ (restrictions on ε follow from the unitarity of the representation, see figure 1). Matrix elements of the supplementary series IRs are expressed via the so-called torus function [44].

An invariant dispersion with respect to $SO(2, 1)$ transformations can be written as

$$\Delta \boldsymbol{J}^2 = \langle \hat{J}_\mu \hat{J}^\mu\rangle\langle \hat{J}_\mu\rangle\langle \hat{J}^\mu\rangle = (\Delta J^0)^2 - (\Delta J^1)^2 - (\Delta J^2)^2. \qquad \text{(A28)}$$

It has the value $j(j+1) - m^2$ on the states $|jm\rangle$. At a given j CS minimize the absolute value of the dispersion (A28). For CS of the discrete series $\Delta J^2 = j$, and for the principal series $\Delta J^2 = -1/4 - \lambda^2/4 - \varepsilon^2$.

We now present a short summary of IRs studied.

For single-valued unitary IRs of $SO(2, 1)$ the angular momentum projection m is integer, for single-valued IRs of $SU(1, 1)$ it is integer or half-integer. For multivalued unitary IRs the projection m can take any real values. Here we find an essential difference from the Lorentz group in four dimensions, for unitary representations of the group this projection is always integer or half-integer. That is connected with the existence of the non-Abelian compact subgroup $SU(2) \sim SO(3)$. Representations of the discrete series $T_j^{\pm}(g)$ of $SU(1, 1)$ at real, integer and half-integer $j < -1/2$ are single valued and have the highest and lowest weights $m = \pm j$. Representations of the principal series $T_{j,\varepsilon}(g)$, $j = -1/2 + \mathrm{i}\lambda$, $-1/2 < \varepsilon \leqslant 1/2$, are single-valued at $\varepsilon = 0$ and at $\varepsilon = 1/2$. At $\varepsilon \neq 1/2$ representations are irreducible and have neither highest nor lowest weights; at $\varepsilon = 1/2$ the representation is split in two ones: $T_{j,1/2}^{-}(g)$ with the highest weight $m = -1/2$ and $T_{j,1/2}^{+}(g)$ with the lowest weight $m = 1/2$.

Now we have to make some technical remarks. As follows from our consideration, representatives of all non-equivalent finite-dimensional and unitary IRs of $SU(1, 1)$ can be

Table A1. Unitary and finite-dimensional IRs of $SU(1,1)$.

Series	S	ζ	s-v or m-v
Finite-dimensional: T_S	$S \geqslant 0$, integer or half-integer	$S-n$, $n \leqslant 2S$	s-v
Discrete:	$S < -1/2$		s-v at $S=-1-n/2$
T_S^+		$-S+n$	
T_S^-		$S-n$	
Principal:			
$T_{S,\varepsilon}$, $-1/2 < \varepsilon \leqslant 1/2$	$S=-1/2+\mathrm{i}\lambda/2$	$\varepsilon \pm n$	s-v at $\varepsilon = 0, 1/2$
$T_{-1/2,1/2} = T^+_{-1/2} \oplus T^-_{-1/2}$,			
$T^+_{-1/2}$	$S=-1/2$	$1/2+n$	s-v
$T^-_{-1/2}$	$S=-1/2$	$-1/2-n$	s-v
Supplementary:	$-1/2 < S < 0$		
$T_{S,\varepsilon}$, $\lvert\varepsilon\rvert < \lvert S\rvert$		$\varepsilon \pm n$	s-v at $\varepsilon = 0$
$T_S^+(\varepsilon = S)$		$\varepsilon + n$	m-v
$T_S^-(\varepsilon = -S)$		$\varepsilon - n$	m-v

constructed in the space of functions on only two complex variables v_1 and v_2. At the same time, studying the left GRR (4.4) of the $M(2,1)$ group, it is convenient to use functions on the elements $z_1, \bar{z}_2$ of the first column of the matrix Z. In such a space the spin generators (4.7) are reduced to the form

$$\hat{S}^0 = (1/2)(z_1\partial/z_1 - \bar{z}_2\partial/\bar{z}_2) \qquad \hat{S}^1 = (\mathrm{i}/2)(z_1\partial/\bar{z}_2 + \bar{z}_2\partial/z_1)$$
$$\hat{S}^2 = -(1/2)(z_1\partial/\bar{z}_2 - \bar{z}_2\partial/z_1). \tag{A29}$$

In fact, after the renotation $z_1 \to v_1$, $\bar{z}_2 \to v_2$ they go over to the generators (A4).

The IRs of $SU(1,1)$ are summarized for the case of spin operators in table A1. We denote the eigenvalue of $\hat{S}^0$ as ζ and the eigenvalue of $\hat{\boldsymbol{S}}^2$ as $S(S+1)$, that corresponds to the renotation $j \to S$, $m \to \zeta$. The parameter n in table A1 is integer and $n \geqslant 0$; s-v or m-v signify single-valued or multivalued IR respectively.

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